Basic thoughts:
Stars shine steadily - changes generally not seen in history.
Fossil evidence: Sun must be around for billions of years.
- They must be very stable for most of their lives.
  * But they radiate energy, so they must evolve.

Stability: HSE

Assume: Spherical, non-rotating, non-magnetic, isolated star...

If it is stable, then no net accelerations:
  (Internal motions average out.)

Mass inside of \( r \) is:

\[
M(r) = \int_0^r 4\pi r^2 \rho(r) \, dr
\]

(Alternately \( \frac{dM}{dr} = 4\pi r^2 \rho \),
  mass continuity.)
Consider the force on a small cylindrical element.

\[ F_p(r+dr) = -P(r+dr)dA \]

\[ F_g = -\rho g dr dA \]

\[ F_p(r) = P(r)dA \]

\[ \sum F = ma = 0 = -P(r+dr)dA + P(r)dA - \rho g dr dA \]

Newton's law: 

\[ \frac{P(r+dr) - P(r)}{dr} = -\rho g = \frac{dp}{dr} \]

Equation of hydrostatic equilibrium:

\[ \frac{dp}{dr} = -\rho g \]  This is the second equation of stellar structure!

Note that \(|g| > 0, \rho \geq 0\), so \(\frac{dp}{dr} \leq 0\)

Pressure gradient balances gravity.
Alternate view - take a global, not local approach

look at perturbations to $W$ - HSE will represent an extremum.

Consider total energy

First, gravitational potential

From Newton, we knew $\Omega = -\frac{Gm_1 m_2}{r}$

A convention is negative

Think of this as $-\Omega$ is required to separate the masses to infinity

$\Omega = 0$ is when masses are infinitely far away

Consider how tightly bound a shell at the surface is

$$d\Omega = -\frac{GM_\star}{r} dM_\star$$
For the whole star, we have

\[ \Omega = - \int_0^M \frac{GM(r)}{r} \, dM \]

\[ \Omega = \frac{GM}{R} \]

\[ \Omega \approx O(1) \]

where \( q \sim O(1) \) and depends on the distribution of the mass in the star.
Ex: constant density

\[ \mathrm{d}M = \frac{4}{5} \pi r^2 \rho \mathrm{d}r \quad \rho = \text{constant} \]

\[ \Omega^2 = -\frac{4\pi}{5} \int_0^R \frac{GM(r)}{r} r^2 \rho \mathrm{d}r \]

\[ M(r) = \frac{4}{3} \pi r^3 \rho \]

\[ = -\frac{(4 \pi)^2}{3} G \rho \int_0^R r^4 \mathrm{d}r \]

\[ = -\frac{(4 \pi)^2}{3} G \rho \frac{R^5}{5} \]

\[ = -\left(\frac{4\pi}{3} \rho R^3\right)^2 \frac{3}{5} G = -\frac{3}{5} \frac{GM^2}{R} \]

This can be thought of as a lower limit.
What about kinetic energy?

We have both internal (microscopic) and macroscopic (turbulence, convection, ...)

Consider macroscopic to be negligible, then

\[ W = \int_{M_{\text{total}}}^{M_{\text{specific internal}}} EdM = U + \mathcal{Q} \]

for equilibrium, we want \( W \) to be a stationary point.

Consider adiabatic motions, infinitesimal

\[ (\delta W)_{\text{ad}} = 0 = (\delta U)_{\text{ad}} + (\delta \mathcal{Q})_{\text{ad}} \]
\[ U \rightarrow U + \delta U = U + \delta \int_M E dM = U + \int_M \delta E \, dM \]

We are following a Lagrangian description.

First law:

\[ dQ = 0 = dE + \rho d(\frac{1}{r}) \]

\[ \delta E = - \rho \delta (\frac{1}{r}) = + \rho \left( \frac{1}{r^2} \right) \delta r \]

\[ \text{where } \rho = \frac{dM}{4\pi r^2 dr} \quad \text{continuity eq.} \]

\[ \frac{dM}{d\left( \frac{4}{3} \pi r^3 \right)} \]

Consider \( \rho + \delta \rho \) — assuming spherical symmetry, we have only radial motions:

\[ \rho + \delta \rho = \frac{dM}{d\left( \frac{4}{3} \pi (r + \delta r)^3 \right)} = \frac{dM}{d\left( \frac{4}{3} \pi r^3 \right) + d\left( \frac{4}{3} \pi r^3 \delta r \right)} \]

\[ = \rho \left( 1 - \frac{1}{1 + \frac{d(4\pi r^2 \delta r)}{d\left( \frac{4}{3} \pi r^3 \right)}} \right) \approx \rho - \rho \frac{d(4\pi r^2 \delta r)}{d\left( \frac{4}{3} \pi r^3 \right)} \]

\[ \approx \delta \rho \]
\[ \delta E = \frac{\dot{\rho}}{\gamma^2} \delta \rho = - \frac{\dot{\rho}}{\rho} \frac{d(4\pi r^2 \delta r)}{d \left( \frac{4}{3} \pi r^3 \right)} \]

and \( (\dot{\varepsilon}_U)_{dd} = - \int \frac{\dot{\rho}}{\rho} \frac{d(4\pi r^2 \delta r)}{dM/M} \, dM \)

\[ = - \int \dot{\rho} \frac{d(4\pi r^2 \delta r)}{dM} \, dM \]

Boundary conditions: \( \delta r (\partial r) = 0 \) = 0 — no motion at center (spherically symmetric)

\( \delta r = r (M(r) = M) = 0 \) — surface (zero BC on \( r \))

Integrate by parts

\[ \delta U = - \int \dot{\rho} \frac{d(4\pi r^2 \delta r)}{dM} \, dM = - \int d \left( \frac{\dot{\rho}}{\rho} \frac{4\pi r^2 \delta r}{dM} \right) \, dM \]

\[ + \int \frac{dP}{dM} (4\pi r^2 \delta r) \, dM \]

\[ \therefore \delta U = \int \frac{dP}{dM} (4\pi r^2 \delta r) \, dM \]
Now for the potential,

$$\phi \rightarrow \phi + S \phi = - \int \frac{GM(\tau)}{r + \epsilon r} \, d\tau$$

expand $$\epsilon r \ll r$$

$$\sim - \int \frac{GM(\tau)}{r} \, d\tau + \int \frac{GM(\tau)}{r^2} \epsilon r \, d\tau$$

$$\therefore (S\mathbb{1})_{dd} = \int \left[ \frac{dP}{dM(r)} \right] \frac{4\pi r^2}{2} + \frac{GM(\tau)}{r^2} \int \epsilon r \, d\tau = 0$$

This is what we want

only way to be zero generally is for

$$\frac{dP}{dM(r)} = - \frac{GM(\tau)}{4\pi r} + \int L \text{ Lagrangian form of HSE}$$

(Your book guides you to explore $$S^2 \mathbb{1}$$ in problem 1, 11)
Vital Theorem

Your text takes a particle–based approach — it considers the motion of each particle, makes some constructs, and then talks about equilibrium.

We want to make a statement about the energy of a stable star.

Start with HSE (= stability)

\[
\frac{dp}{dM} = - \frac{GM}{4\pi r^4}
\]

multiply by volume (\(V = \frac{4}{3}\pi r^3\)) and integrate

\[
\int_{0}^{P(E)} V \, dp = - \frac{1}{3} \int_{0}^{M} \frac{GM \, dM}{r} = \frac{4}{3} + \frac{1}{3} \Omega^2
\]
integrate by parts

\[ \int_0^r V \text{d}P = V \text{d}P \bigg|_0^r - \int_0^r P \text{d}V \]

if we consider the full star, then

\[ P(r) = 0, \quad V(0) = 0 \]

using \( \text{d}V = \frac{\text{d}m}{\rho} \)

\[ \begin{vmatrix} -3 \int_0^M \frac{P}{\rho} \text{d}M = \Omega \end{vmatrix} \]

Vital theorem

Your book has a moment-of-inertia-like term

\[ \frac{d^2 \mathcal{I}}{dt^2} \rightarrow 0 \text{ for stability} \]
Integrate by parts

\[ \int_0^\infty V dP = \left[ \frac{P(r)}{V} \right]_0^\infty - \int_0^\infty \frac{\dot{P}(r)}{V} dV \]

If we consider the full star, then

\[ P(\infty) = 0, \quad V(0) = 0 \]

using \( dV = \frac{dm}{\rho} \)

\[
-3 \int_0^\infty \frac{P}{\rho} dM = \Omega \]

\[ \text{Vinal theorem} \]

Your book has a moment-of-inertial-like term

\[ \frac{d^2 I}{dt^2} \rightarrow 0 \text{ for stability} \]
HSE: \[ \frac{dP}{dr} = \rho g \]

\[ \frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4} \quad \text{(Lagrangian form)} \]

\[ \frac{dM(r)}{dr} = 4\pi r^2 \rho \quad \text{mass conservation} \]

\[ \int_0^M \frac{P}{\rho} dM = -\frac{1}{3} \Omega \quad \text{Virial theorem} \]

\[ \Omega = -\int_0^M \frac{GM(r) dM}{r} \approx -q \frac{GM_*^2}{R_*} \]
Ex: Is spherical symmetry ok?

Sun rotates about once per 27 days

\[ \omega = \frac{2\pi}{P} = \frac{2\pi}{27,804,600 s} = 2.7 \times 10^{-6} \text{ s}^{-1} \]

KE relative to gravitational binding energy

\[ \frac{M\omega^2 R^2}{GM^2/R} = \frac{\omega^2 R^3}{GM} \]

\[ = \frac{(2.7 \times 10^{-6} \text{ s}^{-1})^2 (7 \times 10^{10} \text{ cm})^3}{6.67 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2} \cdot 2 \times 10^{33} \text{ g}} \]

\[ = 1.9 \times 10^{-5} \]
Applications of the Virial Theorem

Global energetics

8-law equation of state

$$P = (\gamma - 1) \rho E$$
(popular in hydrodynamics)

For an ideal gas \( \gamma = \frac{c_p}{c_v} \)  
(note: doesn't need to be monotonic)

for monatomic, \( \gamma = \frac{5}{3} \)

$$E = \frac{3}{2} \frac{P}{\rho}$$  
(should look familiar)

for radiation or completely relativistic Fermi gas, \( \gamma' = \frac{4}{3} \)

Now:

$$3 \int_0^M \frac{P}{\rho} \, dM = -\frac{52}{3 \, \text{total internal energy}}$$

using \( \gamma \)-law

$$3 (\gamma - 1) \int_0^M E \, dM = \frac{52}{3 \, \text{total internal energy}} = 3 (\gamma - 1) U$$

notice that the only thing only \( U \) if \( \gamma = 1 \)
so Virial Theorem says
\[ 3(y-1)U + \Omega = 0 \]

Total energy is \[ W = U + \Omega \rightarrow U = W - \Omega \]

\[ 3(y-1)(W-\Omega) + \Omega = 0 \]
\[ 3(y-1)W - (3y-4)\Omega = 0 \]

and we have
\[ W = \frac{3y-4}{3(y-1)} \Omega \]

Stability requires that we be bound \((W < 0)\)

Since \(\Omega < 0\), we need \( \frac{3y-4}{3(y-1)} > 0 \) or \( y > \frac{4}{3} \)

We will see in some circumstances we get \( y \rightarrow \frac{4}{3} \)
(collapse)
Kelvin–Helmholtz timescale

- We'll see this is important when considering star formation.
- Also useful for energy arguments in powering Sun.

Starting with \( W = \frac{\varepsilon y - 4}{3(y-1)} \mathcal{L} \)

If we consider the star to collapse a bit, then \( W \) changes

\[
\Delta W = \frac{\varepsilon y - 4}{3(y-1)} \Delta \mathcal{L} \quad \text{(we take \( y = \) constant)}
\]

Assume HSE maintained (collapse is slow)

\[
\Delta \mathcal{L} = \frac{GM^2}{R^2} AR < 0 \quad \text{(since \( AR < 0 \))}
\]

Energy is liberated.\(^1\)

* How much of \( \Delta \mathcal{L} \) is radiated and how much heats?

\[
\Delta U = -\frac{\Delta \mathcal{L}}{3(y-1)} \quad \text{(Virial theorem again)}
\]

For \( y = \frac{5}{3} \), \( \Delta U = -\frac{1}{2} \Delta \mathcal{L} \)

\( \frac{1}{2} \) radiated, \( \frac{1}{2} \) goes into heating.
Look at what this says

- Star collapses a bit
- A U increases — this means the star gets hotter
- M decreases — only $\frac{1}{2}$ of $\Delta M$ went into W, the other $\frac{1}{2}$ was lost!
- $\frac{1}{2} \Delta M$ is radiated away.

Loss of energy = hotter star!

This is effectively a negative specific heat
Ex: gravitational lifetime of the Sun

for \( r = \frac{5}{3} \), \( \frac{1}{2} \) of the energy from contraction can be radiated.

Presently, \( \Omega \sim \frac{GM^2}{R} \)

so amount of energy radiated to date is

\[
E_{\text{rad}} \sim \frac{1}{2} \frac{GM^2}{R}
\]

The present luminosity of the Sun is \( L = 4 \times 10^{33} \text{ erg/s} \)

\[
t_{\text{shine}} \sim \frac{E_{\text{rad}}}{L} = \frac{1}{2} \frac{GM^2}{LR}
\]

↑ Kelvin-Helmholtz timescale

\[
= \frac{1}{2} \frac{6.67 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2} \cdot (2 \times 10^{38} \text{ g})^2}{7 \times 10^{10} \text{ cm} \cdot 4 \times 10^{33} \text{ erg/s}}
\]

\[
= 4.7 \times 10^{14} \text{ s} = 1.5 \times 10^7 \text{ yr}
\]
Ex: What is the chemical lifetime of the Sun?

Assume Sun completely H

How many $H$ atoms?

$$N_H = \frac{M_0}{m_H} = \frac{2 \times 10^{33} \text{ g}}{1.67 \times 10^{-24}} \sim 10^{57}$$

What’s a typical energy you can get from chemical reactions?

$\sim 1$ eV (maybe up to 10 eV)

$$E_{\text{chemical}} = 10^{57} \text{ eV} - 1.6 \times 10^{-12} \text{ erg/eV} = 1.6 \times 10^{45} \text{ erg}$$

Chemical lifetime?

$$t \sim \frac{E_{\text{chemical}}}{10^{-12}} = \frac{1.6 \times 10^{45} \text{ erg}}{4 \times 10^{33} \text{ erg/s}}$$

$$= 4 \times 10^{12} \text{ s} = 4 \times 10^7 \text{ yr}$$

$\sim 10^4 \text{ yr}$
Virial of — stellar temperature

Assume ideal, monatomic gas (pretty good for Sun)

\[ E = \frac{3}{2} n k T = \frac{3}{2} \rho k T \]

\[ \rho = \text{density} \]
\[ \frac{1}{m} = N_A \]

\[ M = \text{mean molecular weight/ion} \]
\[ \sim 1 \text{ for stellar mix} \]

\[ U = \int E dN \]

\[ = \frac{3}{2} \rho k T \mu m \]
\[ V = \frac{3}{2} \rho k T \mu m \]

\( \rho V = M \)

**Virial Theorem**: \[ 3(k-1)U + S = 0 \]

\[ k = \frac{5}{3} \]
\[ S = -\frac{1}{2} S \]

Assume uniform density,

\[ S = -\frac{3}{5} \frac{GM^2}{R} \]

\[ \frac{3}{2} \rho k T \mu m = \frac{3}{10} \frac{GM^2}{R} \rightarrow T = \frac{GM \mu m}{5kR} \]

\[ = \frac{6.67 \times 10^{-8} \text{ dyn cm}^2 \text{ g}^{-2} \cdot 2 \times 10^{24} \text{ g} \cdot 1.67 \times 10^{-24} \text{ g}}{5 \cdot 1.88 \times 10^{-16} \text{ erg/K} \cdot 7 \times 10^{10} \text{ cm}} = 4.6 \times 10^6 \text{ K} \]
or compactly, we see \( T \propto M^{2/5} \rho^{1/3} \)

alternatively, HSE

\[
\frac{dP}{dm} = \frac{-GM}{4\pi r^4}
\]

this says that \( P \sim \frac{GM^2}{4\pi R^4} \sim \frac{p k T}{\mu m m} \sim \frac{M k T}{\mu m m R^3} \)

\[
\therefore \quad T \sim \frac{G \mu m m}{4\pi k R} \left( \text{different constant, but same scaling} \right)
\]

\[
T \propto \frac{1}{R} \sim \rho^{1/3}
\]

* book has figure of \( T \) in log \( T \)-log \( y \) plane

Note this is the "average" temperature (whatever that means)

\[
\text{This is high enough to ionize atoms (what is this \( T \) in eV?)}
\]

\[
KT = 1.38 \times 10^{-16} \text{erg/k} \cdot 4 \times 10^6 \text{K} \sim 6 \times 10^{-10} \text{erg}
\]

\[
\times 1.6 \times 10^{-12} \text{eV/erg/eV} = \approx 400 \text{ eV}
\]
Constant density solar model

\[ \rho = \text{constant implies } \frac{M^*}{R^*} = \frac{M(r)}{r^2} \]

Lagrangian HSE:

\[ \frac{dP}{dM} = -\frac{GM(r)}{4\pi r^4} = -\frac{GM^*}{4\pi R^*} \left( \frac{M(r)}{M^*} \right)^{-\frac{4}{3}} \]

BC: \( M(r=0) = 0 \)
\( M(r=R) = M^* \)

\[ \int_{P_c}^P \left( \frac{GM^*}{4\pi R^*} \left( \frac{M(r)}{M^*} \right)^{-\frac{4}{3}} \right) dM \]

\[ P - P_c = -\frac{GM^*}{4\pi R^*} \left( \frac{M(r)}{M^*} \right)^{\frac{2}{3}} \left. \right|_0^M(r) \]

\[ = -\frac{3GM^*}{8\pi R^*} \left( \frac{M(r)}{M^*} \right)^{\frac{2}{3}} \]

Now if \( M(r) = M^* \), then \( P = 0 \), so

\[ P_c = \frac{3GM^*}{8\pi R^*} \]

\( \text{turns out this is a lower limit on } P_c \), since \( \rho \) always decreases with \( r \)

and \( \rho = P_c \left( 1 - \left( \frac{M(r)}{M^*} \right)^{\frac{2}{3}} \right) = P_c \left( 1 - \left( \frac{r}{R} \right)^2 \right) \)

Evaluating finds \( P_c \approx 10^{15} \text{ dyn/cm}^2 \) in the Sun.
Molecular weight example.

We see this come in as

\[ P = n k T \]

\( \rho \) = density

by definition, \( n \) is # of particles / volume.

Here we need the total # of particles, regardless of what they are.

\[ n = \frac{\rho}{\bar{m}} \]

\( \bar{m} \) = average particle mass

(notice that \( \bar{m} = \mu m_0 \))

\( \mu \) = mean molecular weight

---

Now consider completely ionized He.

What is \( \bar{m} \)?

\[ \bar{m} = \frac{1}{2} \left( 4m_u + 2m_e \right) \sim \frac{4}{3} m_u \] so \( \mu = \frac{4}{3} \)

so the electron mass is negligible, but their # matters a lot!
Alternately, \(P = n_kT = n_{\text{ions}}kT + n_{e}kT\)

charge neutrality, \(n_{e} = Zn_{\text{ions}}\) (for multiple species w/ diff Z's there will be a sum here)

\[P = (Z+1)n_{\text{ions}}kT\]

\[
\rho = A_m n_{\text{ions}} + m_en_{e} \sim A_m n_{\text{ions}}
\]

mass of ion

\[
\therefore n_{\text{ions}} \sim \frac{\rho}{A_m}
\]

and \[P = \frac{(Z+1)}{A_m} \rho kT\]

\[
\therefore \mu = \frac{A}{Z+1} = \frac{4}{3} \text{ for the ionized}
\]
What about \( \mu \) generally?

Consider a gas of atoms, ions, and electrons together.

\[ n = n_e + n_i \]

\[ P = P_e + P_i = \frac{p k T}{\mu m_0} = \frac{p k T}{\mu_0 m_0} + \frac{p k T}{\mu_i m_i} \]

\[ \frac{1}{M} = \frac{1}{M_0} + \frac{1}{\mu_i} \]

\[ n_{i,k} = \frac{\rho X_k}{A_k m_i} \quad \text{here } X_k \text{ is mass fraction,} \]

\[ n_i = \sum_k n_{i,k} \]

\[ \rho = \sum_k X_k \rho_k, \quad \sum_k X_k = 1 \]

\[ \frac{1}{M} = \sum_k \frac{X_k}{A_k} \quad \text{(note, we implicitly neglect } m_e \ll m_i) \]

Total mean molecular weight of ions
What about electrons?

Identify nuclei with proton #, \( Z_k \), mass #, \( A_k \)

\[
\rho_{\text{total}} = \sum A_k m_u n_{i,k} + \sum Z_k y_k m_e n_{e,k}
\]

but charge neutrality says

\[
n_{e,k} = y_k Z_k n_{i,k} \quad n_t = \frac{\rho_{i,k}}{A_k m_u} = \frac{x_k \rho}{A_k m_u}
\]

Ionization fraction

\( 0 \leq Z_p \leq 1 \)

\[
n_e = \sum n_{e,k} = \sum \frac{x_k}{A_k} \frac{\rho}{m_u} y_k Z_k
\]

\[
= \frac{\rho}{m_u} \sum y_k \frac{x_k Z_k}{A_k} = \frac{\rho}{m_e m_u}
\]

\[
\equiv \frac{1}{m_e} \quad \text{mean molecular weight per free electron}
\]

\[
n = \frac{\rho}{m m_u} = n_{i} + n_e = \frac{\rho}{m m_u} + \frac{\rho}{m e m_u}
\]
24.

Notation

\[ X = \text{mass fraction of } H \]
\[ Y = \text{mass fraction of } \text{He} \]
\[ Z = \text{mass fraction of metals} \]

\[ X + Y + Z = 1 \]

Assume complete ionization \((y_e = 1 - \text{good inside stars})\)

Assume \(Z \ll 1\)

\[ M_e^{-1} = X + \frac{1}{2} Y \approx X + \frac{1}{2} (1 - X) = \frac{1 + X}{2} \]

\[ \therefore M_e \sim \frac{2}{1 + X} \quad \text{inside stars} \]

\[ M_i^{-1} = X + \frac{Y}{4} \approx X + \frac{1}{4} (1 - X) = \frac{1 + 3X}{4} \]

\[ M_i \sim \frac{4}{1 + 3X} \]

\[ M^{-1} = \frac{1}{M_i} + \frac{1}{M_e} = \frac{1 + 3X}{4} + \frac{1 + X}{2} = \frac{3 + 5X}{4} \]

\[ M \sim \frac{4}{3 + 5X} \]
Now we can evaluate the central $T$ in the constant density model

$$p_0 = \frac{3GM_*^2}{8\pi R_*^4} = \frac{\rho KT}{\mu m_u} = \frac{3M_*}{4\pi R_*^3} \frac{KT}{\mu m_u}$$

$$\Rightarrow T = \frac{1}{2} \frac{GM_*}{R_*} \frac{\mu m_u}{K} \sim 1.2 \times 10^{7} \mu \left( \frac{M}{M_\odot} \right) \left( \frac{R}{R_\odot} \right)^{-1}$$

This is better than the Virial average b/c we used a real pressure profile.