Main assumption: local thermodynamic equilibrium (LTE) (not necessarily true in atmospheres)

At every point in the star, we can use a single temperature/thermodynamic state to describe the material.

Compare photon mean free path

$$\lambda_\gamma = \frac{1}{n \sigma} = \frac{1}{K \rho}$$

- this describes the absorbers (opacity (cm$^2$/g))
- just to mess with us, some books call $n \tau = K$ (i.e. no $\rho$)!

The opacity will contain a lot of atomic physics and details of scattering.

We'll see later that electron scattering dominates, with $K \sim 1$ cm$^2$/g

Then for $\rho \sim \langle \rho_0 \rangle \sim 1$ g/cm$^3$, we have

$$\lambda_\gamma \sim 1 \text{ cm}$$

Aside: think about how long it will take a photon to random walk from the interior to the surface of the Sun!
2. The natural thing to compare to is the pressure scale height.

For an isothermal ideal gas, we can see from HSE that

\[ P = P_0 e^{-x/H} \]

The scale height, \( H = \frac{1}{\frac{d}{dx} \ln P} = \frac{P}{\frac{P_0}{P_0 g}} \) into HSE

In the Sun, consider the constant density model we derived earlier:

\[ P = P_0 \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \]

\[ \frac{d \ln P}{dr} = \frac{P_0}{P} \frac{2r}{R^2} = \frac{2r}{R^2} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]^{-1} \]

so \[ H = \frac{R^2}{2r} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \]

Consider \( H \left( \frac{R}{2} \right) = \frac{R^2}{R} \left[ 1 - \left( \frac{1}{2} \right)^2 \right] = \frac{3}{4} R \)

so \( H \sim R \gg \lambda \phi \)

This supports our statement that we are in LTE.
Recall a distribution function
\[ n(p) \, d^3x \, d^3p = \# \text{ of particles with momentum } p \text{ in volume } d^3x \]

So far we've been since \( n \) as number density (cm\(^{-3}\))
We'll introduce \( N \) as specific number density
\[ N = n/p \]

Then we can define the chemical potential as
\[ \mu_i = \left( \frac{\partial e}{\partial N_i} \right)_{s,v} \]
(Here I am using \( e \) as your text uses \( E \) - specific internal energy)

Chemical equilibrium says that
\[ \sum_i \mu_i \, dN_i = 0 \]
(where \( N_i \) can change due to, ex, reactions)

Note: not all \( dN_i \) are independent - usually when one goes up, another goes down.

Note: it can be argued that \( \mu_g = 0 \) (photons) since their \# is not conserved

We'll use this later when we deal w/ ionization...
General distribution function

\[
\nu(p) = \frac{1}{V} \frac{g}{V^2} \frac{1}{\mathbb{I} \mu + \mathbb{I} E_0 + \mathbb{I} \varepsilon(p) V T} \pm 1
\]

Phase space
Momentum volume

Reference energy

\[+\text{ for fermions (1/2 spin)}\]
\[-\text{ for bosons (0 or integer spin)}\]

From this, number density is simply

\[
n = \int \nu(p) \, 4\pi p^2 \, dp \quad \text{(cm}^{-3}\text{)}
\]

Integration over a sphere
Integrate over all momentum space

Generally, \( \varepsilon(p) = \left( p^2 c^2 + m^2 c^4 \right)^{1/2} - mc^2 \)

For \( pc \ll mc^2 \),

\[
\varepsilon(p) = mc^2 \left( \frac{p^2 c^2}{m^2 c^4} + 1 \right)^{1/2} - mc^2
\]

\[\sim mc^2 \left( 1 + \frac{1}{2} \frac{p^2 c^2}{m^2 c^4} \right) - mc^2 = \frac{1}{2} \frac{p^2}{m} \]

For \( pc \gg mc^2 \),

\[
\varepsilon(p) = pc \left( 1 + \frac{m^2 c^4}{p^2 c^2} \right)^{1/2} - mc^2 \sim pc \left( 1 + \frac{1}{2} \frac{m^2 c^4}{p^2 c^2} \right) - mc^2
\]

\[\sim pc - mc^2 + \frac{1}{2} \frac{m^2 c^4}{pc} \sim pc \]
5. We'll need velocity, from Hamiltonian mechanics,

\[ v = \frac{\partial \mathbf{E}}{\partial \mathbf{p}} \]

Then pressure is

\[ P = \frac{1}{3} \int_{\eta} \mathbf{n}(\mathbf{p}) \mathbf{p} \cdot \mathbf{v} 4\pi \mathbf{p}^2 \, d\mathbf{p} \]

and internal energy is

\[ \mathbf{E} = \int_{\eta} \mathbf{n}(\mathbf{p}) \mathbf{E}(\mathbf{p}) 4\pi \mathbf{p}^2 \, d\mathbf{p} \]

\[ \text{motivation of } P \text{ integral } \quad (\text{C&B O \S 10.2}) \]

Consider a cylinder w/ gas, which elastically collide. When colliding w/ an end, \( \Delta \mathbf{p} \) is entirely in \( \hat{x} \)

\[ \Delta \mathbf{p} = \frac{2 \mathbf{p}_x \hat{x}}{\hat{x}} \quad \text{change in momentum} \]
Now the force on the wall is just

\[ F_{\text{wall}} = - \frac{\Delta p}{\Delta t} = \frac{2p_x}{\Delta t} \hat{x} \]

equal \& opposite

The time between collisions (to either end \& back) is

\[ \Delta t = 2 \frac{\Delta x}{v_x} \]

So \( F_{\text{wall}} = \frac{p_x v_x}{\Delta x} \hat{x} \)

If we average over particles, we need

\[ \langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_z \rangle = \frac{1}{3} p \cdot v \]

no preferred direction

:. the average force / particle is

\[ f(p) = \frac{1}{3} \frac{p v}{\Delta x} \]

If \( N(p) \hat{p} \) is the distribution function integrated over space (so just \# of particles w/ \( p \) in \( p, p + \Delta p \)),

Then the total force from all particles w/ momentum \( p \) is

\[ \Delta F(p) = f(p) N(p) d^3p = \frac{1}{3} \frac{N(p)}{\Delta x} p v d^3p \]
If we divide by the area of the wall, \( A \), we get a pressure

\[ dP = \frac{dF(t)}{A} = \frac{1}{3} \frac{N(p)}{AV} \cdot p \cdot V \cdot d^3p \]

But since \( n(p) \, d^3p \) is the # of particles/volume with momentum \( p \)

\[ n(p) \, d^3p = \frac{N(p)}{AV} \, d^3p \]

so

\[ dP = \frac{1}{3} n(p) \, d^3p \cdot p \cdot V \]

Integrating over all momenta,

\[ P = \frac{1}{3} \int n(p) \cdot p \cdot V \cdot \frac{4\pi p^2 \, dp}{d^3p} \]
Photon gas \[ \left\uparrow \right\rightarrow \text{two states (independent polarizations)} \]

\[ n(p) = \frac{2}{\hbar^2} \frac{1}{e^{p c/kT} - 1} \]

\( \zeta \) only one energy level \( - E_j = 0 \), and \( \mu_p = 0 \)

We can derive the X \# density — note that we should think about this as an average resulting from interactions w/ matter

\[ n_x = 4\pi \int_0^\infty n(p) p^2 dp \]

\[ = \frac{8\pi}{\hbar^3} \int_0^\infty \frac{p^2}{e^{p c/kT} - 1} \, dp \]

take \( x = \frac{p c}{kT} \), \( dx = \frac{c}{kT} \, dp \)

\[ = \frac{8\pi}{h^3} \left( \frac{kT}{c} \right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} \]

\[ = 2 \zeta(3) \sim 2(1.202, \ldots) \]

\( \uparrow \) zeta function

\[ \therefore n_x = 2\pi \zeta(3) \left( \frac{2kT}{\hbar c} \right)^3 \sim 20.3 \ T^3 \ cm^{-3} \]
The radiation pressure is
\[ P_{\text{rad}} = \frac{1}{8} \int_0^\infty 4\pi p^2 \nu n(p) \, dp \]

now \( \nu = c \) and \( E = h\nu = pc \rightarrow p = \frac{h\nu}{c} \)
\[ = \frac{4\pi}{8} \int_0^\infty c \left( \frac{h\nu}{c} \right)^3 \left\{ \frac{2}{h^2} \frac{1}{e^{h\nu/kT} - 1} \right\} \frac{h}{c} \, d\nu \]
\[ = \frac{8\pi}{3} \frac{h}{c^3} \int_0^\infty \frac{x^3 \, dx}{e^{x} - 1} \]

\[ x = \frac{h\nu}{kT} \quad dx = \frac{h}{kT} \, d\nu \]
\[ = \frac{8\pi}{3} \frac{h}{c^3} \left( \frac{kT}{h} \right)^4 \int_0^\infty \frac{x^3 \, dx}{e^x - 1} = \frac{8\pi}{3} \left( \frac{kT}{h} \right)^4 \frac{\pi^4}{15} = \frac{1}{3} \alpha T^4 \]

\( \alpha \) is the radiation constant, \( \alpha = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4} \text{ erg/cm}^3 \)

finally, \( E = \int_0^\infty 4\pi p^2 E(p) n(p) \, dp = 3P_{\text{rad}}, \) since \( E(p) = pc \)

\( \therefore \) this obeys a \( 8^-\text{law EOS}, \) w/ \( \gamma = \frac{4}{3} \) (radiation)
If we see the integrand of our pressure energy integral,

\[ u_0 \, dv = \frac{8\pi h v^3}{c^2} \frac{dv}{e^{h v / k T} - 1} \]

\[ = \frac{4\pi}{c} B_0(T) \, dv \]

we see that the energy density/unit frequency is just \( \propto \) the Planck function.
For an ideal gas, we start with

\[ n(p) = \frac{1}{\frac{\hbar}{\pi}} \frac{g}{e^{-\mu/kT + E_0/kT} / kT + 1} \]

We start by asserting that \( \frac{\mu}{kT} \ll -1 \)

In this case, \( e^{-\mu/kT} \) will dominate over the \( \pm 1 \) term, so

\[ n(p) \sim \frac{g}{\frac{\hbar}{\pi}} e^{\mu/kT} e^{-E_0/kT} e^{-p^2/2mkT} \]

and the density is

\[ n = \frac{4\pi g}{\hbar^3} \int_0^{\infty} p^2 e^{\mu/kT} e^{-E_0/kT} e^{-p^2/2mkT} dp \]

\[ = \frac{4\pi g}{\hbar^3} e^{\mu/kT} e^{-E_0/kT} \int_0^{\infty} p^2 e^{-p^2/2mkT} dp \]

\[ \begin{align*}
X &= \frac{p^2}{2mkT} \\
&= \frac{1}{2mkT} \int_0^{\infty} x e^{-x} dx
\end{align*} \]
\[ x = \frac{p}{\sqrt{2mKT}} \]

\[ dx = \frac{dp}{\sqrt{2mKT}} \]

\[ n = \frac{4\pi g}{h^3} e^{\frac{WKT}{kT}} e^{-E_0/kT} (\frac{2mKT}{\pi})^{\frac{3}{2}} \int_0^\infty x^2 e^{-x^2} dx \]

\[ \frac{1}{2} \Gamma\left(\frac{3}{2}\right) \]

\[ \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \sqrt{\pi} \]

then

\[ n = \frac{\pi^{\frac{3}{2}} g}{h^3} e^{\frac{WKT}{kT}} e^{-E_0/kT} (\frac{2mKT}{\pi})^{\frac{3}{2}} \]

so

\[ e^{\frac{WKT}{kT}} = \frac{\pi g h^3}{(2m\pi KT)^{\frac{3}{2}}} e^{-E_0/kT} \]

our requirement that \( e^{\frac{WKT}{kT}} \ll 1 \) means \( \frac{n}{T^{\frac{3}{2}}} \) needs to be small

(with all the constants put in place)

If \( e^{\frac{WKT}{kT}} \) is not small, corrections will appear—we will not consider these
Now,

\[ P = \frac{4\pi}{3} \int_0^\infty p^4 v n(p) \, dp \]

Take \( v = \frac{p}{m} \)

\[ = \frac{g}{3} \cdot \frac{4\pi}{h^3} \frac{1}{m} \int_0^\infty p^4 \frac{e^{\mu/kT} - e^{-\mu/kT}}{e^{E_0/kT} - e^{-E_0/kT}} e^{-p^2/2mkT} \, dp \]

Take \( \chi = \frac{p^4}{2mkT} \)

\[ dx = \frac{1}{\sqrt{2mkT}} \, dp \]

\[ P = \frac{g}{3} \cdot \frac{4\pi}{h^3} \frac{1}{m} e^{\mu/kT} - E_0/kT (2mkT)^{\frac{3}{2}} \int_0^\infty x^4 e^{-x^2} \, dx \]

\[ = \frac{1}{2} \cdot \frac{1}{\Gamma\left(\frac{5}{2}\right)} \]

\[ \Gamma\left(\frac{5}{2}\right) = \frac{3}{4} \sqrt{\pi} \]

\[ P = \frac{g}{3} \cdot \frac{4\pi}{h^3} \frac{1}{m} e^{\mu/kT} - E_0/kT (2mkT)^{\frac{3}{2}} \frac{3}{4} \sqrt{\pi} \cdot \frac{1}{2} \]

\[ = g \frac{\pi^{\frac{3}{2}}}{m} (2mkT)^{\frac{3}{2}} e^{\mu/kT} e^{-E_0/kT} \frac{1}{2} \frac{1}{h^3} \]

Substituting in our expression of \( e^{\mu/kT} \),

\[ P = \frac{g}{2} \frac{1}{h^3} \frac{\pi^{\frac{3}{2}}}{m} (2mkT)^{\frac{3}{2}} e^{-E_0/kT} \int \frac{n h^3}{g(2mkT)^{\frac{3}{2}}} e^{E_0/kT} \]

\[ = n k T \]
Fermi EOS

e^-, p, n all have spin 1/2 \rightarrow g = 2

We'll look at electron degeneracy

We take $E_0 = mc^2$

$$n = \frac{8\pi}{h^3} \int_0^{\infty} \frac{p^2 dp}{e^{\frac{\mu - mc^2 + E_{\text{kin}}}{kT}} + 1}$$

for fermions

and $E(p) = mc^2 \left[ \sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1 \right]$

$$\nu = \frac{dE}{dp} = \frac{1}{2}mc^2 \left(1 + \left(\frac{p}{mc}\right)^2\right)^{-1/2} \frac{p}{mc} \frac{1}{mc}$$

$$= \frac{p}{m} \left(1 + \left(\frac{p}{mc}\right)^2\right)^{-1/2}$$

($\sim \frac{p}{m} \left[1 - \frac{1}{2}\left(\frac{p}{mc}\right)^2\right]$ for $p \ll mc$)
Doing the general integral is hard. Lots of approximation exist. We won't delve into all of these, but consider:

complete degeneracy

\[ F(E) = \frac{1}{e^{(E-(\mu-mc^2))/kT}+1} \]

consider \( T \to 0 \)

if \( E > (\mu-mc^2) \) then we get \( \frac{1}{e^{\infty}+1} \sim 0 \)

if \( E < (\mu-mc^2) \) then we get \( \frac{1}{e^{-\infty}+1} \sim 1 \)

so the distribution function becomes a step function

Fermi energy: \( E_F = \mu-mc^2 \)

\[ F(E) = \begin{cases} 1 & E < E_F \\ 0 & E > E_F \end{cases} \]

\( n(p) = \frac{2}{h^2} F(p) \)

All states \( \forall E < E_F \) are completely filled

Fermi momentum: \( x_F = \frac{p_F}{mc} \) then \( E_F = mc^2 \left[ (1+x_F^2)^{1/2} - 1 \right] \)

Note that \( m_F = E_F + mc^2 \) is the total energy of the most energetic particle in the system
last time we headed

in LTE, we can describe the photons, e-, and ions/nuclei as having the same f

We will express the pressure as

\[ P = P_{\text{gas}} + P_{\text{rad}} + P_{\text{elec}} \]
\[ E = E_{\text{gas}} + E_{\text{rad}} + E_{\text{elec}} \]

In normal stars, electrons are not degenerate, and we can just lump them together with ions, as.

\[ P_{\text{gas}} = P_{\text{ion}} + P_{\text{elec}} \]

It will be convenient to define

\[ \beta = \frac{P_{\text{gas}}}{P} \]

Note that for mixtures, while \( P \) and \( E \) are additive, various adiabatic exponents (like \( \Gamma \), you'll consider in your homework is not)

Your book gives expressions for specific heats and various adiabatic exponents—we'll see them as needed.
so far:

\[ n(p) \] is the distribution function
- \# density w/ momentum \( p \) in \( d^3p \)

\[
n(p) = \frac{1}{\hbar^2} \frac{g}{e^{(-\mu + E_0 + E(p))/kT} + 1}
\]

\(*\) degeneracy of states

\( g \)

reference \( \uparrow \) + for fermions
- for bosons

\# density

\[
n = \int n(p) \, 4\pi p^2 \, dp \quad (\text{cm}^{-3})
\]

pressure

\[
p = \frac{1}{3} \int n(p) v_p \, 4\pi p^2 \, dp \quad (\text{dyn/cm}^2)
\]

energy density

\[
E = \int E(p) \, 4\pi p^2 \, dp \quad (\text{erg/cm}^3)
\]

\[ E(p) = (p^2 c^2 + mc^4)^{\frac{1}{2}} - mc^2 \rightarrow \left\{ \begin{array}{ll}
\frac{p^2}{2m} & \text{NR} \\
\frac{p}{mc} & \text{ER}
\end{array} \right. 
\]

\[ v = \frac{\partial E}{\partial p} \]
radiation:

\[ n_\gamma = 2\pi g_0 (3) \left( \frac{2k}{\nu c h} \right)^3 \nu^3 \]
\[ P_\gamma = \frac{1}{8} a T^4 \]
\[ E_\gamma = 3 P_\gamma \]

ideal gas:

\[ P = n k T \]
\[ E = \frac{3}{2} n k T \]

degenerate electrons

limit \( T \to 0 \)

\[ n(p) = \frac{2}{h^3} \begin{cases} 1 & \varepsilon < \varepsilon_F \\ 0 & \varepsilon > \varepsilon_F \end{cases} \]

\[ \varepsilon_F = \mu - mc^2 \]

Fermi momentum:

\[ \varepsilon_F = mc^2 \left[ \left( 1 + x_F^2 \right)^{1/2} - 1 \right] \quad \text{w/} \quad x_F = \frac{p_F}{mc} \]

\( \mu_F \) is the energy (including \( mc^2 \)) of the most energetic particle in the system.
How do we find \( E_F, x_F, \) or \( \mu_F? \)

We constrain things based on the # density \( \rho \):

\[
N = \frac{8\pi}{h^2} \left( \int_0^{p_F} p^2 \, dp \right) = \frac{8\pi}{3} \left( \frac{mc}{h} \right)^3 \int_0^{x_F} x^2 \, dx = \frac{8\pi}{3} \left( \frac{mc}{h} \right)^3 x_F^3
\]

We only need to integrate to \( p_F \), since \( F(p) = 0 \) beyond that.

\[
\therefore \quad x_F = \left( \frac{3}{8\pi} \right)^{\frac{1}{3}} \frac{h}{mc} n^{\frac{1}{3}}
\]

Now \( n_e = \frac{\rho}{\mu_e m_0} \) for electrons, and

\[
\rho_e = \frac{8\pi}{3} \left( \frac{mc}{h} \right)^2 x_F^3 = \frac{\rho}{\mu_e m_0}
\]

\[
\frac{\rho}{\mu_e} = \frac{8\pi m_0}{3} \left( \frac{mc}{h} \right)^3 x^2 \approx 10^6 \text{ g/cm}^3 x^3
\]

Note that \( x = x_F - 1 \) is about the dividing line between non-relativistic and extremely relativistic.

In a WD, the above shows this occurs @ \( \rho/\mu_e \approx 10^6 \text{ g/cm}^3 \)

(for NS, \( m_e \to m_n \) which is \( \approx 10^3 \) larger, so \( \rho \approx 10^8 \) larger)
Also note: there is no $T$ dependency—later we’ll look at some #s to see how a finite $T$ changes things.

What about $p$?

$$p = \frac{1}{2} \int p v n(p) \, dp$$

$$= \frac{4\pi}{3} \frac{1}{\hbar^2} \frac{1}{m} \int_0^{p_F} \frac{p^4}{(1 + (p/mc)^2)^{1/2}} \, dp$$

$$x = \frac{p}{mc}$$

$$= \frac{8\pi}{3} \frac{1}{m \hbar^3} (mc)^5 \int_0^{x_F} \frac{x^4}{\sqrt{1 + x^2}} \, dx \equiv A f(x)$$

$$A = \frac{\pi}{3} \left( \frac{mc}{\hbar} \right)^3 mc^2$$

$$f(x) = x(2x^2 - 3)(1 + x^2)^{1/2} + 3 \sinh^{-1} x$$

Note to use this, you first find $x = x_F$ from the # density expression, and then evaluate $p$ w/ that $x$. 
to do this integral

\[ I = \int_0^{x_F} \frac{x + t}{\sqrt{1 + x^2}} \, dx \]

\[ u = \sqrt{1 + x^2} \]

\[ u^2 = (1 + x^2) \]

\[ 2u \, du = 2x \, dx \]

\[ \therefore \, dx = \frac{u \, du}{x} = \frac{u \, du}{\sqrt{u^2 - 1}} \]

\[ I = \int \frac{(1 + x_F^2)^{\frac{1}{2}}}{1} \frac{(u^2 - 1)^{\frac{3}{2}}}{u} \, du \]

\[ = \int \frac{(1 + x_F^2)^{\frac{1}{2}}}{(u^2 - 1)^{\frac{3}{2}}} \, du = \frac{\sqrt{u^2 - 1}^3}{4} - \frac{3\sqrt{u^2 - 1}^{\frac{1}{2}}}{8} + \frac{3}{8} \ln \left( \frac{u + \sqrt{u^2 - 1}}{u} \right) \]

\[ \text{[Spiegel 19.55]} \]

\[ = \frac{1}{8} \left\{ 2(1 + x_F^2)^{\frac{1}{2}} x_F^3 - 3(1 + x_F^2)^{\frac{3}{2}} x_F + 3 \ln \left( \frac{1 + x_F^2 + x_F}{\sqrt{1 + x_F^2 - x_F}} \right) \right\} \]

\[ I = \frac{1}{8} \left\{ x_F \left( 2x_F^2 - 3 \right) (1 + x_F^2)^{\frac{1}{2}} + 3 \sinh^{-1} x_F \right\} \]

\[ = \frac{1}{8} f(x) \]
\[ E \text{ is similar} \]

\[ E = \int n(p) E(p) 4\pi p^2 \, dp \]

\[ = 4\pi \hbar^2 \int_0^{p_F} p^2 \, mc^2 \left[ (1 + (\frac{p}{mc})^2)^{\frac{1}{2}} - 1 \right] \, dp \]

\[ x = \frac{p}{mc} \]

\[ = \frac{8\pi}{\hbar^3} \, mc^2 \left( \frac{mc}{\hbar} \right)^3 \int_0^{x_F} x^2 \left[ (1 + x^2)^{\frac{1}{2}} - 1 \right] \, dx \]

\[ = A \, g(x) \]

same \( A \), \( A = \frac{\pi}{3} \left( \frac{mc}{\hbar} \right)^3 mc^2 \)

\[ g(x) = 8x^3 \left[ (1 + x^2)^{\frac{1}{2}} - 1 \right] - f(x) \]

note: this \( E \) is \( \text{erg/cm}^3 \) (same as \( \text{dyn/cm}^2 \))

Remember: this is in the limit of \( T \to 0 \)

but we did not make any approximations as to whether we are relativistic or non-relativistic... yet.
for hw, you will consider the expansions of these \((f(x) \& g(x))\) in the limits that \(x \ll 1 \& x \gg 1\).

We find:

\[
P_e = \begin{cases} 
A \frac{8}{5} x^5 & x \ll 1 \\
A 2x^4 & x \gg 1 
\end{cases}
\]

Taking

\[
x = \left( \frac{3}{8\pi} \right)^{\frac{1}{3}} \left( \frac{\hbar}{mc} \right)^{\frac{1}{3}} \nu_e = \left( \frac{3}{8\pi} \right)^{\frac{1}{3}} \left( \frac{\hbar}{mc} \right) \left( \frac{p}{\mu m_e} \right)^{\frac{1}{3}}
\]

we have:

\[
P_e \propto \begin{cases} 
\left( \frac{\nu_e}{v} \right)^{5/3} & \text{non-relativistic} \\
\left( \frac{p}{\nu_e} \right)^{4/3} & \text{extreme relativistic}
\end{cases}
\]

and the same \(\nu\)-dependence for \(E\).

Also we can show that:

\[
\frac{E_e}{P_e} = \begin{cases} 
\frac{3}{2} \chi \left( \gamma = \frac{5}{2} \right) & x \ll 1 \\
\frac{3}{5} \chi \left( \gamma = \frac{4}{3} \right) & x \gg 1
\end{cases}
\]
Consider a WD — degenerate electrons provide most of the support.

Virial theorem:
\[ 3(2 - 1) U = - S \]

Assume constant \( \rho \)

\[ S L = - \frac{3}{5} \frac{G M^2}{R^*} \]

\[ U = V E_e = \left( \frac{4\pi}{3} R^3 \right) \left( \frac{12}{5} A x^5 \right) \]

Take \[ \frac{\rho}{M_e} = B x^5 \]

\[ B = \frac{8\pi m_v}{3} \left( \frac{m_e c}{h} \right)^3 \]

\[ x = \left( \frac{1}{B} \frac{\rho}{M_e} \right)^{\frac{1}{5}} = \left( \frac{1}{B} \frac{3}{4\pi} \frac{M^*_e}{R^*_e} \right)^{\frac{1}{5}} \]

Then, \( w/ \ y = \frac{5}{3}, \)

\[ U = - \frac{1}{2} S L \]

\[ \frac{4\pi}{3} R^*_e \frac{18}{5} A \left( \frac{1}{B} \frac{3}{4\pi} \frac{M^*_e}{R^*_e} \right)^{\frac{5}{3}} = \frac{3}{2} \frac{G M^2}{R^*} \]
\[
\frac{32\pi}{3} \left( \frac{3}{4\pi} \right)^{5/3} \frac{A}{G} \frac{5/3}{b^{5/3}} M_a^{5/3} \frac{1}{R_a^{5/3}} = \frac{M^*}{R^*}
\]

\[M_a^{5/3} \sim \frac{1}{R_a} \quad \text{(for constant density)}\]

This means that more massive WDs are smaller.

Plugging in #s finds

\[\frac{M^*}{M_0} \sim 10^{-6} \left( \frac{R}{R_0} \right)^{-3} \left( \frac{2}{\mu_0} \right)^{5}\]

This implies WDs have radii \(\sim\) Earth's radius.

Later, we'll do better by deriving a second-order ODE for the structure of a polytrope gas.

This result is for non-relativistic WDs!
What about the relativistic case?

\[ \gamma = \frac{4}{3} \]

\[ U = -52 \]

\[ \left( \frac{4}{3} \pi R_*^3 \right) A \cos^4 x = \frac{3}{5} \frac{GM_*^2}{R_*} \]

and

\[ x = \left( \frac{1}{B} \frac{\rho}{\mu_e} \right)^{1/3} \]

then

\[ 8 \pi R_*^2 A \left( \frac{1}{B} \frac{3 M_*}{4 \pi R_*^2 \mu_e} \right)^4 = \frac{3}{5} \frac{GM_*^2}{R_*} \]

notice that \( R_* \) cancels out!

We are left with

\[ M_*^{2/3} = \frac{40 \pi}{3} A \left( \frac{3}{4 \pi B} \right)^{4/3} \left( \frac{1}{\mu_e} \right)^{4/3} \rightarrow M_* \sim \text{const} \left( \frac{1}{\mu_e} \right)^2 \]

We can evaluate this, but b/c of the approximations we made, it is not going to be the physical value.

Later we'll see how to solve this self-consistently.

The real result is \( M_* \sim 1.4 M_\odot \left( \frac{2}{\mu_e} \right)^2 \)
So far, we assumed $T \to 0$ (complete degeneracy)

What about finite $T$? When does $T$ begin to matter (lift the degeneracy)?

For low $T$, most of the electrons are still as they were w/ complete degeneracy.

If we measure energy as $kT$ and $kT \ll E_F$, then only the electrons already near $E_F$ can escape to the region $E > E_F$

We can use $E_F \sim kT$ as the transition region between degeneracy and non-degeneracy

Since $E_F \sim \frac{1}{2} mc^2 x^2_F = \frac{1}{2} mc^2 \left( \frac{1}{B} \frac{P}{M_0} \right)^{\frac{2}{3}} \sim kT$

(N-R)
\[
\frac{p}{\mu_e} = \frac{(\frac{1}{2}mc^2)^{3/2}}{B} \frac{p}{\mu_e} = k^{3/2} \ T^{3/2}
\]

\[
\frac{p}{\mu_e} = B \left( \frac{2K}{mc^2} \right)^{3/2} T^{3/2} \sim 6 \times 10^{-9} \left( \frac{T}{1k} \right)^{3/2} \text{ g cm}^{-3}
\]

If \( \frac{p}{\mu_e} > 6 \times 10^{-9} \left( \frac{T}{1k} \right)^{3/2} \text{ g cm}^{-3} \), we are degenerate (N-R)

For the relativistic case,

\[
E_F \sim mc^2 \ k_F \sim mc^2 \left( \frac{1}{B} \frac{p}{\mu_e} \right)^{1/2} \sim kT
\]

\[\therefore \frac{p}{\mu_e} = \left( \frac{k}{mc^2} \right)^3 B \ T^3
\]

\[= 4.6 \times 10^{-24} \left( \frac{T}{1k} \right)^3 \text{ g cm}^{-3}
\]

If \( T \) is large, partial degeneracy is attained.

Today we usually do the necessary integrals numerically.
We ignored ionization (so far...)

chemical potential: \( \mu_i = \left( \frac{\partial e}{\partial N_i} \right)_{s, v} \)

\[ \text{this is specific } \# \text{, } \mathcal{N} = \frac{n}{\rho} \]

chemical equilibrium

\[ \sum \mu_i \, dN_i = 0 \quad \text{this takes into account changes in } \# \text{ by reactions} \]

typically when \( i \) \( N_i \) goes up, another falls
Chemical potential:

First law becomes

\[ \text{d}e = T \text{d}s - P \text{d}V + \frac{1}{T} \text{d}\mu N \]  
(all differentials are specific, i.e., per unit mass)

from Shu, \( \text{d}s > 0 \) implies that particles flow from region of high \( \mu \) to low \( \mu \)

here we see that the chemical potential is the change in free energy resulting from the addition of a particle
Consider ionization

\[ H^+ + e^- \rightarrow H^0 + \chi_{HH} \]
\[ \text{Ionization potential, } 13.6 \text{eV} \]

LTE # of \( H^+, e^-, \) and \( H^0 \)

Chemical equilibrium implies

\[ \sum_i \mu_i v_i = 0 \]
\( \mu_i \) coefficients in the reaction

then

\[ \mu^e + \mu^+ - \mu^0 = 0 \text{ for equilibrium (energy is not changing in equilibrium)} \]

Note we know from ideal gas that

\[ n_e = \frac{2 \cdot (2\pi m_e kT)^{3/2}}{\hbar^3} e^{\mu^e/kT} - \frac{E^e/kT}{e} \]

\[ n^+ = \frac{(2\pi m_p kT)^{3/2}}{\hbar^3} e^{\mu^+/kT} - \frac{E^+/kT}{e} \]

Now, we'll take \( E^e = E^+ = 0 \) (this is our reference — the just ionized state)

then \( E^0 = -\chi_{HH} \)

\[ n^0 = \frac{2 \cdot (2\pi m_p kT)^{3/2}}{\hbar^3} e^{\mu^0/kT} + \chi_{HH} kT \]
The constraint that $\mu^e + \mu^+ - \mu^0 = 0$ can be realized by forming

$$\frac{n^+ n^e}{n^0} = \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\frac{m}{kT}} e^{\left(\mu^+ + \mu^e - \mu^0\right)/kT}$$

...equilibrium requires that

$$\frac{n^+ n^e}{n^0} = \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\frac{m}{kT}}$$

where $m = \frac{m_e m_p}{m_e + m_p} \sim m_e$ was used.

define

$$n = n^0 + n^+ \quad \text{(nuclei)}$$

charge neutrality,

$$n^e = n^+$$

define ionization fraction as

$$y = \frac{n^+}{n^0 + n^+} = \frac{n^+}{n} = \frac{n^e}{n}$$

$$\frac{n^+ n^e}{n^0} = \frac{y^2 n}{1-y} = \left(\frac{2\pi m_e kT}{\hbar^2}\right)^{3/2} e^{-\frac{m}{kT}}$$

This allows you to solve for ionization fraction.
Useful discriminant: degeneracy parameter
\[ \eta = \frac{\mu - m_e c^2}{kT} \]
\[ \eta > 1 \text{ is degenerate} \]

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Coulomb effects

- Compare Coulomb potential between two ions \( \frac{Z^2 e^2}{q \ell_{\text{separation}}} \)

Then define \( a \) via \( \frac{4\pi}{3} a^3 \sim \frac{1}{n_I} \) (Wigner-Seitz sphere)

\[ \Gamma_c > 1 \text{ means Coulomb effects become important} \]

\( \Gamma_c \gg 1 \) crystalization sets in (important for WD cooling)

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Radiation important when \( \frac{1}{8} a \Gamma c^4 \sim \frac{p eT}{\mu \text{cm}} \)

Pressure ionization: in dense gas, electric field of one atom disrupts neighboring atoms — ionization sets in.

\( a \sim \text{Bohr radius}, \ \rho \sim 1 \text{ g cm}^{-3} \text{ for ionization} + \text{interactive with helium gas} \)