Navier-Stokes Equations

- We’ll follow Choudhuri, Ch. 5 pretty closely
- Shu and some other physics fluids books (e.g. Tritton) give some nice discussions as well.
Navier-Stokes Equations

- We have

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} &= 0 \\
\frac{\partial (\rho v_j)}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i v_j) + \frac{\partial P_{ij}}{\partial x_i} &= \frac{\rho}{m} F_j \\
\frac{\partial (\rho e)}{\partial t} + \frac{\partial}{\partial x_i} (\rho e v_i) + P_{ij} \Lambda_{ij} + \frac{\partial q_i}{\partial x_i} &= 0
\end{align*}
\]

- We wrote:

\[P_{ij} = p \delta_{ij} + \pi_{ij}\]

Note: some sources put a ‘-’ here instead of a ‘+’
The Euler equations represent the most basic system of fluid equations.

There is quite a bit of physics that is neglected, including viscosity, thermal diffusion, and body forces.

- All of these can be important in astrophysical flow.

On the microscopic scale, the molecules making up the fluid experience friction against one another, providing some dissipation, which we call the viscosity.

- Viscous stresses oppose relative movements between neighboring fluid particles.
- These motions redistribute momentum.
Navier-Stokes Equations  
(Tritton Ch. 1, 2, 5)

- Consider a fluid moving all in the same direction, with a speed that varies with height, $v = v(y)$.

- Across any arbitrary plane, perpendicular to $y$, a stress will act.

- The faster fluid above the plane drags the lower fluid forward. Similarly, the slower fluid below drags the upper fluid back.

- Equal and opposite forces will act, with the stress proportional to the velocity gradient, giving a force/unit area of

$$\tau = \mu \frac{\partial v_x}{\partial y}$$
The net force on a fluid element is the small difference of the viscous stresses on either side of it.

Per unit area perpendicular to $y$, the forces are

$$\mu \left( \frac{\partial v_x}{\partial y} \right)_{y+\delta y} \quad \text{and} \quad -\mu \left( \frac{\partial v_x}{\partial y} \right)_y$$

acting in the $x$-direction between our plates.

The net force on the fluid element is then

$$\left( \mu \left( \frac{\partial v_x}{\partial y} \right)_{y+\delta y} - \mu \left( \frac{\partial v_x}{\partial y} \right)_y \right) \delta x = \frac{\partial}{\partial y} \left( \mu \frac{\partial v_x}{\partial y} \right) \delta x \delta y$$
Navier-Stokes Equations

- Note that this force would appear as:

\[
\frac{\partial (\rho v_x)}{\partial t} + \nabla \cdot (\rho \mathbf{v} v_x) + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( \mu \frac{\partial v_x}{\partial y} \right)
\]

- Again, this is in the case for the prescribed flow in the previous figure.
- Notice that the viscous term is greater the stronger the velocity gradient in the perpendicular direction.

- In this case, we’d write:

\[
\pi_{xy} = -\mu \frac{\partial v_x}{\partial y}
\]
Navier-Stokes Equations

- This force needs to be accounted for in the momentum equation, yielding
  \[ \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) + \nabla p = \mu \nabla^2 \mathbf{v} \]

  - A corresponding term is added to the energy equation.

- This is actually a great simplification, which assumes that the stress is in the plane of the surface.
  - This is the case that Newton considered.

- Fluids that obey this type of viscous term are called **Newtonian fluids**
  - For many fluids, this is a fine approximation
To see why our simple expression

$$\pi_{xy} = -\mu \frac{dv_x}{dy}$$

is not general, consider a rigidly rotating fluid.

Here, $\frac{\partial v_x}{\partial y}$ is not zero

- But shear stress, $\pi_{xy}$, has to be zero, since there are no relative motions inside the fluid.
- More thorough argument (blackboard) finds

$$\pi_{ij} = -\mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right)$$

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Navier-Stokes Equations
(Choudhuri Ch. 5)

- Our velocity equation is then:

\[
\rho \frac{Dv_i}{Dt} + \frac{\partial p}{\partial x_i} = \rho f_i + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) \right]
\]

- Often, we take the viscosity to be constant, giving:

\[
\rho \frac{Dv_i}{Dt} + \frac{\partial p}{\partial x_i} = \rho f_i + \mu \left( \nabla^2 v_i + \frac{1}{3} \frac{\partial}{\partial x_i} \nabla \cdot \mathbf{v} \right)
\]

- And in many cases, we take the flow to be weakly compressible

\[
\rho \frac{Dv_i}{Dt} + \frac{\partial p}{\partial x_i} = \rho f_i + \mu \nabla^2 v_i
\]
We define the *kinematic viscosity* as $\nu = \mu/\rho$, and write:

$$\frac{D v_i}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = f_i + \nu \nabla^2 v_i$$

- Many people refer to this form as the Navier-Stokes equation.
Navier-Stokes Equations

- Note: this is now a second order equation.

\[ \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) + \nabla p = \mu \nabla^2 \mathbf{v} \]

- This behavior of the solution dramatically, and if viscosity is important, different methods are needed.

- An additional boundary condition is needed now.
  - We can specify both the normal and tangential (no slip) velocity at walls (with ideal fluids, we could not)
  - Your text makes a nice observation about how fan blades collect dust

- Computational techniques for solving this type of equation will differ from the inviscid case.
Pipe Flow

- Consider flow through a pipe
  
  - We’re using cylindrical coordinates
  - Flow is setup by a pressure gradient over the length, \( l \), of the pipe
    - No radial velocity: \( \mathbf{v} = v \hat{z} \)
    - Velocity is uniform in \( z \) direction: \( \mathbf{v} = v(r) \hat{z} \)
  
  - Assume steady flow: \( \frac{Dv}{Dt} = 0 \)

- We get a 2\textsuperscript{nd} order ODE that we can solve (blackboard):
  \[
  v(r) = \frac{1}{4} \frac{\Delta p}{\mu l} (a^2 - r^2)
  \]
Pipe Flow

- This is sometimes called *Poiseuille flow*
- We can find the mass flux through the pipe:

  \[ Q = \int_0^a \rho v(r)r\,dr\,d\theta = \frac{\pi}{8} \frac{\Delta p}{\nu l} a^4 \]

  - Experimentally, if we measure the mass flux, we can find the kinematic viscosity of the fluid.
  - Note: this expression only holds if the flow is not too fast (low Reynolds number)
Pipe Flow

- Drawings by Reynolds

From Wikipedia: “Water flow observed in a pipe, as drawn by Osborne Reynolds in his best-known experiment on fluid dynamics in pipes. Water flows from left to right in the transparent tube, and dye (represented in black) flows in the middle. The nature of the flow (turbulent, laminar) can be observed easily. These drawings were published in Reynolds’ influential 1883 paper An experimental investigation of the circumstances which determine whether the motion of water in parallel channels shall be direct or sinuous and of the law of resistance in parallel channels.”

This file joins figures 3, 4 and 5 in Reynolds’ 1883 paper marked "received and read" March 15, 1883. Paper was reproduced in his 1901 book "Papers on Mechanical and Physical Subjects, vol.2" which is today published by Forgotten Books.
Viscous Effects

- Back to our velocity equation:

\[
\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} \nu^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}
\]

- Taking the curl:

\[
\frac{\partial \mathbf{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{\omega}) = \nu \nabla^2 \mathbf{\omega}
\]

- This last term doesn’t vanish

- This means that vorticity can be created!
Viscous Effects

- Reynolds number scaling of this gives:

\[
\frac{\partial \omega}{\partial t} - \nabla \times (\mathbf{v} \times \omega) = \frac{1}{\text{Re}} \nabla^2 \omega
\]

- For flow through the pipe, we are laminar for \( \text{Re} < 3000 \), but we become turbulent beyond this

- Consider the limit \( \text{Re} \ll 1 \):

\[
\nabla^2 \omega = 0
\]

  - This can be solved for flow past a sphere (we won’t)
  - Find that sphere experiences a drag opposite its motion

\[
f_D = 6\pi \mu a U
\]

- This is Stokes’s law
- This gets around d’Alembert’s paradox
Low Re Flow

- Because the time-derivatives are small compared to the viscous term, very low Re flow is reversible

See for discussion:

*From Calculus to Chaos, An Introduction to Dynamics* by David Acheson (referenced on that page)
Boundary Layers

- We might think then that for \( \text{Re} \gg 1 \) we could ignore the viscous term \( \rightarrow \) ideal fluid result
  - Experiments show that this is not the case

From
(original source unknown)

PHY 688: Astrophysical Fluids and Plasmas
Karman Vortex Street

(KTover2/Wikipedia)
Boundary Layers

- Experiments show the turbulent drag experienced behind a sphere
  \[ f_D = \frac{1}{2}C_D \pi a^2 \rho U^2 \]
  - Equating this to Stokes’s law
    \[ C_D \sim \frac{12\nu}{aU} = \frac{12}{\text{Re}} \]

- This says that the turbulent drag should decrease linearly with Re, but that’s not seen
Boundary Layers

- For a viscous fluid, we have $\mathbf{v} = 0$ on the surface
  - Normal and tangential
- Large flow velocity should give a large tangential velocity
- This is fixed in a thin *boundary layer*
  - $\mathbf{v} = 0$ at the surface to a large value a short distance away
  - $\nabla^2 \mathbf{v}$ will become large in this layer (and we cannot neglect the viscous term even if $\mu$ is small)
- Note: astrophysical systems typically don’t have solid surfaces, so we don’t consider boundary layers
- There is one major place where viscosity is important: accretion disks
Accretion Disks

- Close binary systems can interact by mass exchange
- Angular momentum conservation: mass doesn’t fall directly onto the surface of companion—disk forms
- Gravitational potential energy released as heat and radiation
  - For compact objects, this can be a significant fraction of rest mass
- Basic theory worked out by Shakura & Sunyaev (1973)
  - We’ll follow Choudhuri’s discussion

Illustration Credit: ESA, NASA, and Felix Mirabel (French Atomic Energy Commission and Institute for Astronomy and Space Physics/Conicet of Argentina)
Accretion Disks

• Consider a disk of negligible mass around a central object with mass \( M \)
  
  – Force balance:

  \[
  \frac{mv^2}{r} = m\Omega^2 r = \frac{GMm}{r^2} \rightarrow \Omega = \left(\frac{GM}{r^3}\right)^{1/2}
  \]

  – Here, angular velocity, \( \Omega \), is called Keplerian velocity

• Note: there is a lot of shear in this disk structure—viscosity should be important
Accretion Disks

- **Shear**: $\Omega \sim r^{-3/2}$
  - Material at greater radius lags behind
- **Effect of viscosity**: 
  - Angular momentum transferred from faster inner regions of disk to slower outer regions
  - Inner material loses angular momentum—spirals inward
- **Viscosity determines the rate of radial inflow (accretion)**, and hence gravitational potential energy release.
Accretion Disks

- We’ll consider the disk dynamics in cylindrical coordinates
  - Assume the disk is thin
  - Ignore the z-velocity: \( \mathbf{v} = v_r \mathbf{\hat{r}} + v_\theta \mathbf{\hat{\theta}} \)
  - Assume azimuthal symmetry: \( \partial/\partial \theta = 0 \)

- Our momentum equation is:
  \[
  \rho \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = \mu \nabla^2 \mathbf{v}
  \]

  - Note: we need to be careful with the derivatives, since the derivative of the unit vectors may not be zero

- The \( \theta \)-component is:
  \[
  \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} \right) = \mu \left( \frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \right)
  \]
Accretion Disks

- Mass continuity is more straightforward:

\[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) = 0 \]

- Note: we assumed that \( \mu = \rho v = \text{constant} \), but this may not be correct, so we’ll figure out the viscous stress another way

- Define surface density:

\[ \Sigma = \int \rho dz \]

- Ignore velocity dependence on \( z \) and integrate our eqs. over \( z \)

\[ \frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) = 0 \]
\[ \frac{\partial (\Sigma r^2 \Omega)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma r^3 \Omega v_r) = G \]

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Accretion Disks

• Notice that the latter equation is an angular momentum equation:

\[
\frac{\partial (\Sigma r^2 \Omega)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma r^3 \Omega \nu_r) = G
\]

– Since

\[
L = mv_\theta r = m\Omega r = (2\pi r dr \Sigma) \Omega r
\]

• This suggests that our RHS is related to a viscous torque, G

\[
G \cdot 2\pi r dr = G(r + dr) - G(r) \rightarrow G = \frac{1}{2\pi r} \frac{\partial G}{\partial r}
\]

– We can write our viscous stress as \( \mu r \, d\Omega/\partial r \) and find an expression for the viscous torque
Accretion Disks

- Final system:

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \Sigma v_r \right) = 0
\]

\[
\frac{\partial (\Sigma r^2 \Omega)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( \Sigma r^3 \Omega v_r \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( \nu \Sigma r^3 \frac{d\Omega}{dr} \right)
\]

- If we assume we are Keplerian, we can equate the time derivatives and find an expression for the velocity:

\[
v_r = - \frac{3}{r^{1/2} \Sigma} \frac{\partial}{\partial r} (\nu \Sigma^{1/2} r^{1/2})
\]

- This is the radial velocity in the disk—note the dependence on viscosity
Accretion Disks

- Our continuity equation becomes:

\[ \frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[ r^{1/2} \frac{\partial}{\partial r} (\nu \Sigma r^{1/2}) \right] \]

- This is a diffusion equation (called the thin disk diffusion equation)
- Note timescale, \( t_{\text{visc}} \sim R^2/\nu \)
- This is the result that Shakura & Sunyaev got
Accretion Disks

- Solution for ring at radius $r_0$

Notice that most of the mass moves inward!
Accretion Disks

• Note: Shakura & Sunyeav showed that a thin disk implies Keplerian
  – This is because the radial pressure gradient becomes unimportant
  – A thick disk would not be Keplerian
• We can find the steady state solution:

\[ \nu \Sigma = \frac{\dot{m}}{3\pi} \left[ 1 - \left( \frac{r_*}{r} \right)^{1/2} \right] \]

  – Mass flow is proportional to viscosity
• It is common to use an \( \alpha \)-viscosity:

\[ \nu = \alpha c_s h \]

  – Viscosity may be due to turbulence or magnetic effects