Rotation
Rotating Flows

• Many astrophysical systems rotate:
  – Rapid stellar rotation can deform a star
  – ISM rotation can help support against collapse

• Differential rotation vs. solid body rotation is important
  – Ex for Choudhuri: spin a bucket of water about the vertical axis
    • Water near the walls moves first—angular velocity varies with \( r \)
    • Shear means that viscous forces will act
    • Eventually viscosity means the water rotates as a solid body
      (uniform angular velocity)
  – In astrophysical systems, viscosity can be weak/negligible:
    differential rotation may be present
  – Forcing can maintain differential rotation
Rotating Flows

• Consider steady axisymmetric rotation

\[ \frac{\partial}{\partial t} = 0 \quad \frac{\partial}{\partial \theta} = 0 \quad v_r = 0 \]

• Navier-Stokes equation (in cylindrical coords):

\[ - \frac{v_\theta^2}{r} = g_r - \frac{1}{\rho} \frac{\partial p}{\partial r} \]

  – We already looked at this with accretion disks (where pressure gradient was not important)
Rotating Flows

- Consider slow rotating star
  - Pressure gradient and $g$ nearly cancel
  - Small part of $p$ balances centrifugal force
  - Molecular viscosity will be negligible
  - Turbulent diffusion (driven via convection) can be large
    - Angular momentum can be transported
    - May be anisotropic transport
- Stellar models find:
  - Differential rotation sustained w/ anisotropic transport
  - Solid body rotation emerges w/ isotropic transport
Instability

- Consider steady axisymmetric, differential rotation
- Swap fluid rings at $r_0$ and $r_1$
- Look at fluid element originally at $r_0$:
  - Conservation of angular momentum
    \[ l = v_0 r_0 = v_{\text{new}} r_1 \]
    \[ v_{\text{new}} = \frac{r_0}{r_1} v_0 \]
- Stable if this ring wants to return to its original position
Instability

- Ring previously at $r_1$ had centripetal acceleration:

$$\frac{v_1^2}{r_1}$$

- This was provided by the excess of pressure gradient over HSE

- New ring at this position needs centripetal acceleration:

$$\frac{r_0^2 v_0^2}{r_1^3}$$

- Stability: we need this to be less than the original centripetal acceleration → forces to push it back to its initial position

$$\frac{r_0^2 v_0^2}{r_1^3} < \frac{v_1^2}{r_1}$$
Instability

• In terms of angular velocity:

\[ \Omega = \frac{v}{r} \]

- Stability condition:

\[ (\Omega_0 r_0^2)^2 < (\Omega_1 r_1^2)^2 \]

- This is the Rayleigh criterion for stability

\[ \frac{d}{dr} \left[ (r^2 \Omega)^2 \right] > 0 \]
Consider inertial reference frame $C$ and non-inertial frame $C'$

- Separated by vector $l$
- Non-inertial frame is rotating about fixed axis with angular velocity $\Omega$

Fluid element $P$ is at location $r$ in $C$ and $r'$ in $C'$

\[ r = r' + l \]

- In terms of components
  \[ r_i e_i = r'_i e'_i + l_i e_i \]

- $e_i$ are the unit vectors in $C$, $e'_i$ are the unit vectors in $C'$
Rotating Frame

- Total time rate of change:

\[
\frac{D r_i}{D t} e_i = \frac{D r'_i}{D t} e'_i + r'_i \frac{D e'_i}{D t} + \frac{D l_i}{D t} e_i
\]

- Unit vectors in C are fixed
- In C’, unit vectors exhibit circumferential motion (due to rotation)

\[
\frac{D e'_i}{D t} = \Omega \times e'_i
\]

- We have:

\[
\frac{D r}{D t} = \frac{D r'}{D t} + \Omega \times r' + \frac{D l}{D t}
\]
Rotating Frame

• Assume no translation, then:

\[ \mathbf{v} = \mathbf{v}' + \Omega \times \mathbf{r}' \]

- Differentiating this (blackboard):

\[ \frac{D\mathbf{v}}{Dt} = \frac{D\mathbf{v}'}{Dt} + 2\Omega \times \mathbf{v}' + \Omega \times (\Omega \times \mathbf{r}') \]
Rotating Frame

- Our momentum equation goes from:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{f} + \nu \nabla^2 \mathbf{v}
\]

- to

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{f} + \nu \nabla^2 \mathbf{v} - 2\Omega \times \mathbf{v} - \Omega \times (\Omega \times \mathbf{r})
\]
Centrifugal “Potential”

- Consider cylindrical coords with rotation axis in z-direction
  - We can do this without loss of generality

- Our centrifugal term is:
  \[-\Omega \times (\Omega \times \mathbf{r})\]
  - \(\mathbf{r}\) is the distance from the rotation axis
    - Any component in the same direction as \(\Omega\) vanishes in the cross product

- We see that this is identical to:
  \[
  \frac{1}{2} \nabla \left( |\Omega \times \mathbf{r}|^2 \right)
  \]
Centrifugal “Potential”

- Writing our force (gravity) in terms of a potential, we have

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \left( \phi - \frac{1}{2} |\Omega \times \mathbf{r}|^2 \right) + \nu \nabla^2 \mathbf{v} - 2\Omega \times \mathbf{v}$$

- Effective potential:

$$\phi_{\text{eff}} = \phi - \frac{1}{2} |\Omega \times \mathbf{r}|^2$$

- This shows that the centrifugal term is important when it is a significant fraction of the gravitational potential
Rossby Number

- Compare advection to Coriolis force, *Rossby number*:
  \[ Ro = \frac{V^2/L}{\Omega V} = \frac{V}{L\Omega} \]

- Coriolis force is important for \( Ro \lesssim 1 \)

- Note: for differentially rotating bodies, we can pick \( \Omega \) for the rotating frame to be an average
Geostrophic Approximation

- Consider that rotation dominates, flow changes slowly (Dv/Dt is small)
  - Low Rossby number
  - We can also neglect the centrifugal force and viscosity

- Equilibrium:
  
  \[
  - \frac{\nabla p}{\rho} - |g|\hat{e}_r - 2\Omega \times \mathbf{v} = 0
  \]

  - Vertical structure: gravity dominates over Coriolis

  \[
  \frac{\partial p}{\partial r} = -\rho |g|
  \]

- This is our familiar expression for HSE
Geostrophic Approximation

- Horizontal structure:
  \[ \nabla_h p = -2 \rho (\Omega \times \mathbf{v})_h \]
  - ‘h’ subscript indicates lateral gradient

- Since gravity dominates, we expect:
  \[ |\nabla_h p| \ll |\partial p / \partial r| \]

- This system is the geostrophic approximation:
  \[ \nabla_h p = -2 \rho (\Omega \times \mathbf{v})_h \]
  \[ \frac{\partial p}{\partial r} = -\rho |g| \]

  - Notice: for a lateral pressure gradient, it is not the normal velocity, but a transverse velocity that balances it.
Vorticity

- Consider an ideal (no viscosity) incompressible fluid
  - We can write: $\nabla p/\rho = \nabla(p/\rho)$
  - Momentum equation:
    \[
    \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \mathbf{\omega} = -\nabla \left( \frac{p}{\rho} + \frac{1}{2}v^2 + \Phi - \frac{1}{2}|\Omega \times \mathbf{r}|^2 \right) - 2\Omega \times \mathbf{v}
    \]
  - Taking the curl:
    \[
    \frac{\partial \mathbf{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{\omega}) + \nabla \times (\mathbf{v} \times 2\Omega)
    \]
  - Assuming that $\Omega$ is constant:
    \[
    \frac{\partial}{\partial t}(\mathbf{\omega} + 2\Omega) = \nabla \times (\mathbf{v} \times (\mathbf{\omega} + 2\Omega))
    \]
Circulation

• We saw an equation of this form when we last discussed vorticity and proved that:

\[ \frac{\partial \mathbf{Q}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{Q}) \rightarrow \frac{d}{dt} \int_S \mathbf{Q} \cdot d\mathbf{S} = 0 \]

• Therefore:

\[ \frac{d}{dt} \int_S (\omega + 2\Omega) \cdot d\mathbf{S} = 0 \]

  - This is called Bjerknes’s circulation theorem
  - Generalization of the Kelvin circulation theorem
Circulation

- Example:
  - Spread out layer with no vorticity initially
  - Change in $\Omega \cdot dS$ means vorticity with opposite sense develops

Figure 9.1 The spreading of a volume of fluid into a thinner layer in a rotating frame of reference.

from Choudhuri

PHY 688: Astrophysical Fluids and Plasmas
Taylor-Proudman Theorem

- Consider a steady fluid in a rotating frame:
  \[ \nabla \times [\mathbf{v} \times (\omega + 2\Omega)] = 0 \]
  - Assume slow flow, so the vorticity is small compared to \( \Omega \)
    \[ \nabla \times (\mathbf{v} \times \Omega)] = 0 \]
  - Expanding, we find:
    \[ (\Omega \cdot \nabla)\mathbf{v} = 0 \]

- This says that \( \mathbf{v} \) does not change in the direction of \( \Omega \)
  - Slow steady flows in rotating frames are invariant parallel to rotation axis
Ekman Pumping

- Consider stirring a tea cup—the liquid rotates
  - Rotation is fastest at the top of the cup since it is a free surface
  - At the bottom, the base of the cup introduces a drag on the liquid—velocity smaller

- Rotation means that fluid elements experience centrifugal acceleration
  - Rotation profile means force is greatest at the top
  - Direction of force is outward in all cases
Ekman Pumping

- Outward centrifugal force wants to push the water to the sides of the cup
  - Water is incompressible
  - Conservation of mass means that something has to take the place of the displaced water
  - Circulation pattern forms—top is outward because centrifugal force was greatest there
- Stirring a tea cup, this has the effect of pushing tea leaves at the bottom toward the center
Coriolis Force in Astro

- XRBs:
  - Ignition likely begins at a single point
  - Spreads laterally
  - Rapid rotation means Coriolis force confines spread (geostrophic flow)

Fig. 3.13. Initial evolution of a burning hot spot ignited off the equator as seen in a frame rotating with the neutron star. Velocity vectors show the circulation of the fluid induced by the Coriolis forces. The hot spot expands due to burning and drifts west-southwest because of the latitude dependence of the Coriolis force (after Spitkovsky, Levin & Ushomirsky 2002).
Rotation + Gravity

- Rotating self-gravitating bodies are no longer spherical
  - Centrifugal force will lessen the effect of gravity on the equator

![Diagram showing the effect of rotation on self-gravitating bodies.](image-url)
Rotation + Gravity

- Ex: Saturn is about 10% longer across equator than pole to pole

Saturn seen edge-on, as imaged by HST in 1995 (Erich Karkoschka (University of Arizona Lunar & Planetary Lab) and NASA)
Rotation + Gravity

- Centrifugal force makes rotating self-gravitating bodies flatten.
- Solid body rotation, in rotation frame:
  - Fluid is at rest ($v = 0$)
  - Constant density equilibrium:
    \[
    \frac{p}{\rho} + \Phi - \frac{1}{2} |\Omega \times r|^2 = \text{constant}
    \]
- Surface: $p = 0$
  - Taking z-axis to be rotation:
    \[
    \Phi - \frac{1}{2} \Omega^2 (x^2 + y^2) = \text{constant}
    \]
- This can be solved analytically (for $\rho = \text{constant}$)
Rotation + Gravity

- The solution takes on the shape of an ellipsoid:
  \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

- Gravitational potential at any point inside is:
  \[ \Phi = \phi G \rho (\alpha_0 x^2 + \beta_0 y^2 + \gamma_0 z^2 - \chi_0) \]

\[ \alpha_0 = abc \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)\Delta} \]
\[ \beta_0 = abc \int_0^\infty \frac{d\lambda}{(b^2 + \lambda)\Delta} \]
\[ \gamma_0 = abc \int_0^\infty \frac{d\lambda}{(c^2 + \lambda)\Delta} \]
\[ \chi_0 = abc \int_0^\infty \frac{d\lambda}{\Delta} \]

\[ \Delta = \left[ (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \right]^{1/2} \]
Rotation + Gravity

- Requiring this potential to match our surface condition, we find (blackboard):

\[
\left(\alpha_0 - \frac{\Omega^2}{2\pi G \rho}\right) a^2 = \left(\beta_0 - \frac{\Omega^2}{2\pi G \rho}\right) b^2 = \gamma_0 c^2
\]

- Maclaurin showed we can solve this for \( a = b > c \)
  - This is a spheroid
  - Analogy to an ellipse (\( c \) plays the role of the semi-minor axis), we have an eccentricity:

\[
e^2 = 1 - \frac{c^2}{a^2}
\]
Maclaurin Spheroids

- Maclaurin:

\[
\alpha_0 = \beta_0 = \frac{(1 - e^2)^{1/2}}{e^3} \sin^{-1} e - \frac{1 - e^2}{e^2}
\]

\[
\gamma_0 = \frac{2}{e^2} \left[ 1 - (1 - e^2)^{1/2} \frac{\sin^{-1} e}{e} \right]
\]

Note: Choudhuri has a typo in this expression—see Binney & Tremaine table 2-1
Maclaurin Spheroids

- Surface condition is then:
  \[
  \frac{\Omega^2}{\pi G \rho} = \frac{2(1 - e^2)^{1/2}}{e^3} (3 - 2e^2) \sin^{-1} e - \frac{6}{e^2} (1 - e^2)
  \]

- Rotating star takes on eccentricity \( e \)
- \( e \) increases with rotation (at start)
- For high \( e \), the moment of inertia increases very fast, so \( \Omega \) goes down even though total angular momentum increases
Consequence of Rotation

- Consider a star that is rotating. HSE becomes:

\[ \frac{\nabla p}{\rho} = -\nabla \Phi_{\text{eff}} \]

- Taking the curl:

\[ \nabla p \times \nabla \rho = 0 \]

  - Contours of constant \( p \) and constant \( \rho \) are aligned, and are surfaces of constant \( \Phi_{\text{eff}} \)
  - Ideal gas: \( T \) is constant on equipotentials of \( \Phi_{\text{eff}} \) too

- Oblate spheroid:

  - Contours of \( \Phi_{\text{eff}} \) are closer together at the pole than at the equator
Consequence of Rotation

- Radiation transport is proportional to $\nabla T$
  - Flux is greater at the poles

- Von Zeipel paradox:
  - Nuclear reactions at core provide the energy flux
  - Core doesn’t know what is happening at the surface

- Eddington showed that meridional circulation can get around this (see Kippenhahn & Weigert)
  - Called Eddington-Sweet circulation
  - Differential rotation may also be a way around this