Magnetohydrodynamics
MHD

- MHD uses the one-fluid model of a plasma
  - In a neutral fluid, collisions allow us to treat it under the fluid approximation
  - For a plasma, collisions and the effect of the magnetic field to confine particles enables the fluid behavior
- MHD assumes that charge separation is negligible
- We can model the plasma using an extension of the hydro equations we already developed along with an evolution equation for the magnetic field
Equations of MHD

- We’ll now assume that transport coefficients are scalars
  - They could be anisotropic in strong fields
  - Isotropy for collision frequency $>$ cyclotron frequency
- We’ll also assume that transport coefficients are spatially constant
- Mass continuity is unchanged:
  \[
  \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
  \]
  - Mass of ions and electrons in plasma are conserved
In the presence of a magnetic field, fluid elements will experience the Lorentz force

- A stationary charge in one reference frame appears to be moving in another frame
- Pure E field in lab frame gives mix of E and B in moving reference frame
Lorentz Force

- Lorentz transformation results (see, e.g., Jackson)
  - Lab frame: $E$, $B$
  - Frame moving with velocity $v$ with respect to lab frame: $E'$, $B'$
  - Transforms:
    
    $$
    E'_\parallel = E_\parallel \\
    B'_\parallel = B_\parallel \\
    E'_\perp = \gamma \left( E_\perp + \frac{v}{c} \times B_\perp \right) \\
    B'_\perp = \gamma \left( B_\perp - \frac{v}{c} \times E \right)
    $$
    
    with
    
    $$
    \gamma = \left( 1 - \frac{v^2}{c^2} \right)^{-1/2}
    $$
Non-Relativistic Limit

- We are interested in non-relativistic flows
  - However, we need to keep $O(v/c)$ terms in order to capture the effect that one frame has pure B but a moving frame has B and E
- To $O(v/c)$, $\gamma = 1$
  - We have:
    \[
    E' = E + \frac{v}{c} \times B
    \]
  - This works because
    \[
    v \times B_{\parallel} = 0
    \]
- Now we need to understand the current (look over Chouduri and the notes by Braithwaite’s notes, linked on our website)
  - Ohm’s law:
    \[
    j = \sigma \left( E + \frac{v}{c} \times B \right) = \sigma E'
    \]
Non-Relativistic Limit

- We’ll focus on highly conducting plasmas
  - This means that since
    \[ \mathbf{j} = \sigma \mathbf{E}' \]
  - For \( \mathbf{j} \) to remain finite, when \( \sigma \to \infty \), we must have \( \mathbf{E}' \sim 0 \)
    \[ |\mathbf{E}| \approx \frac{|\mathbf{v}|}{c} |\mathbf{B}| \]
  - Alternately: highly conducting assumption means that a current can flow to neutralize the rest frame electric field \( \mathbf{E}' \) (see Braithwaite 1.2.1)
- Since \( \mathbf{E} \propto O(v/c) \), we don’t need to include the \( v/c \) term in \( \mathbf{B}' \perp \)
  - Therefore \( \mathbf{B}' \sim \mathbf{B} \) to \( O(v^2/c^2) \)
Non-Relativistic Limit

- Consider Ampère’s circuital law:

\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \]

- Scaling of the displacement current:

\[ \left| \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right| \sim \frac{\mathbf{E}}{\mathbf{B}} \sim \frac{|\mathbf{v}| |\mathbf{E}|}{c |\mathbf{B}|} \]

- Since \(|\mathbf{E}| \sim (|\mathbf{v}|/c) |\mathbf{B}|\) for a highly conducting fluid

\[ \left| \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right| \sim \frac{v^2}{c^2} \]

- We can neglect the displacement current:

\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} \]
Electric Field in MHD

- In MHD, the magnetic field and current density are directly related
  - If we have one, we can get the other
- From Ohm’s law

\[
E = \frac{j}{\sigma} - \frac{v}{c} \times B = \frac{c}{4\pi\sigma} \nabla \times B - \frac{v}{c} \times B
\]

- The electric field is not independent in MHD
- We can derive \( E \) from \( B \)
Electric Field in MHD

- In MHD:
  - E field in rest frame volume of the fluid is always negligible
  - We can find E in any other frame from the B field
  - Velocities are assumed non-relativistic
  - High-conductivity means charge density is low, so we can ignore Maxwell’s

\[ \nabla \cdot \mathbf{E} = 4\pi \rho_e \]
Equations of MHD

- Momentum equation gains the Lorentz force:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla p + \frac{1}{\rho c} \mathbf{j} \times \mathbf{B} + \nu \nabla^2 \mathbf{v}
\]

- We can rewrite this in a few ways:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla \left(p + \frac{B^2}{8\pi}\right) + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi \rho} + \nu \nabla^2 \mathbf{v}
\]

- or

\[
\rho \frac{Dv_i}{Dt} = \rho f_i - \frac{\partial}{\partial x_j} (P_{ij} + M_{ij})
\]

- magnetic pressure
- magnetic tension
- stress tensor
- magnetic stress tensor
Equations of MHD

● In conservative form, this is:

\[
\frac{\partial (\rho v)}{\partial t} + \nabla \cdot \left[ \rho vv + \left( p + \frac{B^2}{8\pi} \right) I - \frac{BB}{4\pi} \right] = 0
\]
Magnetic Stress Tensor

- Magnetic stress tensor:

\[ M_{ij} = \frac{B^2}{8\pi} \delta_{ij} - \frac{B_i B_j}{4\pi} \]

- Just like the diagonals of \( P_{ij} \) are pressure, diagonals of \( M_{ij} \) are magnetic pressure

- Consider field in z-direction:

\[ \mathbf{B} = B \hat{z} \]

\[ M_{ij} = \begin{pmatrix} \frac{B^2}{8\pi} & B^2/8\pi \\ B^2/8\pi & \frac{B^2}{8\pi} - \frac{B^2}{4\pi} \end{pmatrix} \]
Magnetic Stress Tensor

- Here, $M_{zz}$ has the magnetic pressure term + an additional negative term.

- Visually (following Spruit):
  - Surface forces exerted by the field in volume $V$ on its surroundings:
    \[ F_s = \hat{n} \cdot M \]
    - Positive: we are looking at outward force from within fluid element.

- On right face:
  \[
  F_{\text{right},x} = \frac{B^2}{8\pi} - \frac{B_x B_z}{4\pi} = \frac{B^2}{8\pi}
  \]
  \[ F_{\text{right},[y,z]} = 0 \]
  - Outward pressure.

Figure 1.6: Force vectors exerted by magnetic stress on the surfaces of a rectangular box uniform of magnetic field.

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Magnetic Stress Tensor

• On top face:

\[
F_{\text{top},z} = \frac{B^2}{8\pi} - \frac{B_z B_z}{4\pi} = -\frac{B^2}{8\pi}
\]

\[
F_{\text{top},[x,y]} = 0
\]

– Stress is perpendicular to surface
– Inward pointing

• Magnetic pressure alone should point outward, cause fluid element to expand
  – This happens in \(x\) and \(y\) directions here
  – Along the field line, volume contracts

Figure 1.6: Force vectors exerted by magnetic stress on the surfaces of a rectangular box uniform of magnetic field

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Magnetic Stress Tensor

• Along the field lines, stress behaves like negative pressure
  – Analogous to stretching an elastic band
  – Negative stress is a tension

• We can interpret the off-diagonal terms as magnetic tension
Induction Equation

• Starting with Maxwell’s equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$$

– And introducing our electric field:

$$\mathbf{E} = \frac{c}{4\pi \sigma} \nabla \times \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{B}$$

– We get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}$$

• This is the *magnetic induction equation*

• Magnetic diffusivity:

$$\lambda = \frac{c^2}{4\pi \sigma}$$
MHD

- So far we have:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[ \rho \mathbf{v} \mathbf{v} + p^* \mathbf{I} - \frac{\mathbf{B} \mathbf{B}}{4\pi} \right] = 0
\]

\[
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0
\]

- with

\[
p^* = p + \frac{B^2}{8\pi}
\]
Energy Equation

- Choudhuri focuses mainly on incompressible problems
  - The energy equation is not needed there
- It we recall our internal energy evolution, with MHD we pick up an Ohmic heating term:
  \[
  \frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) + p \nabla \cdot \mathbf{v} = \frac{j^2}{\sigma}
  \]
  - This captures the conversion between magnetic energy and thermal energy (heating)
Energy Equation

• We can derive a magnetic energy evolution equation:
  
  - Start with:
    \[
    \frac{\partial B}{\partial t} = \nabla \times (v \times B) + \lambda \nabla^2 B
    \]
  
  - Dot with \( B/(4\pi) \)
  
  - Lot’s of algebra...

  \[
  \frac{\partial}{\partial t} \left( \frac{B^2}{8\pi} \right) + \nabla \cdot \left( v \frac{B^2}{8\pi} \right) = -M_{ij} \frac{\partial v_i}{\partial x_j} - \frac{c}{4\pi \sigma} \nabla \cdot (j \times B) - \frac{j^2}{\sigma}
  \]

• Note: we see the Ohmic heating term that complements the thermal energy
Ideal MHD

- From now forward, we’ll assume ideal MHD
  - Zero thermal conductivity
  - Zero viscosity
  - Zero resistivity (infinite electrical conductivity)
- Our magnetic energy equation becomes:
  
  \[
  \frac{\partial}{\partial t} \left( \frac{B^2}{8\pi} \right) + \nabla \cdot \left( \mathbf{v} \frac{B^2}{8\pi} \right) = -M_{ij} \frac{\partial v_i}{\partial x_j}
  \]
Energy Equation

- We can get the kinetic energy evolution from $\mathbf{v} \cdot \text{momentum}$:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 \right) + \frac{\partial}{\partial x_j} \left( \frac{1}{2} \rho v^2 v_j \right) = -v_i \frac{\partial p}{\partial x_i} - v_i \frac{\partial}{\partial x_j} M_{ij}$$

- Adding internal energy, we can construct total energy evolution:

$$\frac{\partial}{\partial t} \left( \rho e + \frac{1}{2} \rho v^2 + \frac{B^2}{8\pi} \right) + \nabla \cdot \left[ \left( \rho e + \frac{1}{2} \rho v^2 + \frac{B^2}{8\pi} + p \right) \mathbf{v} \right] = -\frac{\partial}{\partial x_j} (v_i M_{ij})$$

- Define total energy as:

$$\rho E = \rho e + \frac{1}{2} \rho v^2 + \frac{B^2}{8\pi}$$

$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot [(\rho E + p) \mathbf{v}] = -\frac{\partial}{\partial x_j} (v_i M_{ij})$$
Energy Equation

- This can also be rewritten as:

\[
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot [(\rho E + p^*) \mathbf{v} - (\mathbf{B} \cdot \mathbf{v}) \mathbf{B}] = 0
\]

- With

\[
p^* = p + \frac{B^2}{8\pi}
\]
Ideal MHD

- Full system of equations is for *ideal MHD*:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[ \rho \mathbf{v} \mathbf{v} + p^* \mathbf{I} - \frac{\mathbf{B} \mathbf{B}}{4\pi} \right] &= 0 \\
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot \left[ (\rho E + p^*) \mathbf{v} - (\mathbf{B} \cdot \mathbf{v}) \mathbf{B} \right] &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) &= 0
\end{align*}
\]
Induction

- We have:

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B} \]

- Note the similarity to the vorticity evolution equation:

\[ \frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{v} \times \omega) + \nu \nabla^2 \omega \]

- We define the magnetic Reynolds number \( R_m \)

\[ R_m = \frac{V B / L}{\lambda B / L^2} = \frac{LV}{\lambda} \]

- As with Re, \( R_m \) is very large (in general) for astrophysical systems
Induction

- Lab vs. astrophysics
  - \( R_m \ll 1 \) for lab systems
    \[
    \frac{\partial \mathbf{B}}{\partial t} \sim \lambda \nabla^2 \mathbf{B}
    \]
  - \( R_m \gg 1 \) for astrophysical systems
    \[
    \frac{\partial \mathbf{B}}{\partial t} \sim \nabla \times (\mathbf{v} \times \mathbf{B})
    \]
- We’ll see that \( \lambda \nabla^2 \mathbf{B} \) can be important in reconnection when \( \nabla \mathbf{B} \) is high
Induction

- Looking at

\[ \frac{\partial \mathbf{B}}{\partial t} \sim \lambda \nabla^2 \mathbf{B} \]

- This is a diffusion equation—magnetic field in a lab plasma decays
  - Ohmic dissipation drives away currents that generate the fields via
    \[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} \]
  - Lab experiments drive $\mathbf{B}$ via a current
Flux Freezing

- Ideal MHD: infinite conductivity (zero resistivity)

\[ \frac{\partial \mathbf{B}}{\partial t} \sim \nabla \times (\mathbf{v} \times \mathbf{B}) \]

- We already saw equations of this form too:

\[ \frac{\partial \mathbf{Q}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{Q}) \quad \rightarrow \quad \frac{d}{dt} \int_{s} \mathbf{Q} \cdot d\mathbf{S} = 0 \]

- Therefore we have for ideal MHD

\[ \frac{d}{dt} \int_{s} \mathbf{B} \cdot d\mathbf{S} = 0 \]

- Magnetic fields move with the fluid (*Alfvén’s theorem of flux freezing*)

This is an analogue to Kelvin’s vorticity theorem
Flux Freezing

- We know a direct relation between $j$ and $B$

$$\nabla \times B = \frac{4\pi}{c} j$$

- In astro, we have an intuitive feel for $B$—it follows $v$
- Since we can visualize $v$, we know what will happen to $B$
- See figure in Choudhuri

- Another consequence:
  - Two fluid elements initially connected by a field line are always connected by the same field line (for ideal MHD)
Magnetohydrostatics

- We want to consider equilibrium configurations, analogous to hydrostatic equilibrium
  
  \[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla p + \frac{1}{4\pi \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} \]

- Static ideal MHD:
  
  \[ \rho \mathbf{f} - \nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \]

- Equation of \textit{magnetohydrostatics}
Magnetohydrostatics

• In contrast to neutral fluids, this solutions are not really static
  – For $v = 0$, we have:
    \[
    \frac{\partial B}{\partial t} = \lambda \nabla^2 B
    \]
  – $B$ will decay eventually—equilibrium will be disturbed
• Labs: $B$ is maintained by driving external current
• Astro: decay is slow ($R_m \gg 1$) so we have approximate equilibrium
• Ignore body forces:
  \[
  \rho f - \nabla p + \frac{1}{4\pi} (\nabla \times B) \times B = 0
  \]
Magnetohydrostatics

• Ignore body forces:

\[ \nabla p = \frac{1}{4\pi} (\nabla \times B) \times B \]

- Pressure must be balanced by magnetic stresses
- Here, \( p \) is *pressure-balanced field*

• Plasma-\( \beta \): gas pressure to magnetic pressure

\[ \beta = \frac{p}{B^2/(8\pi)} \]

- Low-\( \beta \) plasma: gas pressure is negligible
- Field adjusts itself so magnetic stress vanishes (*force-free field*):

\[ (\nabla \times B) \times B = 0 \]
Pressure Balanced Column

• What is the static structure of a column of plasma?

• Start with:

\[ \nabla p = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} \]

• Assume cylindrical symmetry—no variations in \( \theta \) or \( z \)

\[ \mathbf{B} = \mathbf{B}(r) \]

• \( \nabla \cdot \mathbf{B} = 0 \) means \( B_r = 0 \)

\[ \mathbf{B} = B_\theta(r)\hat{\mathbf{r}} + B_z(r)\hat{\mathbf{z}} \]

• Magnetohydrostatic expression becomes (taking \( p = p(r) \)):

\[ \frac{d}{dr} \left( p + \frac{B_\theta^2 + B_z^2}{8\pi} \right) + \frac{B_\theta^2}{4\pi r} = 0 \]
Pressure Balanced Column

- Take magnetic field in plasma to be produced by current
  \[ \mathbf{j} = j(r)\hat{z} \]

- Magnetic field is:
  \[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} \]
  - We cannot have a \( B_z \) component
  \[ \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) = \frac{4\pi}{c} \mathbf{j} \]

- Case I: constant current density \( \mathbf{j} \)
  \[ B_\theta = \frac{2\pi}{c} j r \]
Pressure Balanced Column

- Into magnetohydrostatics equation:

\[ \frac{dp}{dr} + \frac{2\pi}{c^2} j^2 r = 0 \]

- Solution:

\[ p = p_0 - \frac{\pi j^2 r^2}{c^2} \]

  gas pressure at column center

- Pressure calls off with \( r \)—pinching effect (magnetic pressure)

- Pressure falls to zero at radius:

\[ a = \left( \frac{p_0}{\pi} \right)^{1/2} \frac{c}{j} \]
Magnetic Pinching

- Magnetic pinching may be able to confine a plasma
Pressure Balanced Column

- Consider a general $j(r)$
- We want plasma confined to a radius $r = a$
- Total current is (blackboard):
  \[ I = \int_0^a 2\pi r j(r) dr = \frac{c}{2} a B_\theta(a) \]
- Into our magnetohydrostatic equation
  \[ \frac{dp}{dr} = -\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2 B_\theta^2}{8\pi} \right) \]
- Average pressure:
  \[ \bar{p} = \frac{I^2}{2\pi c^2 a^2} \]
Z-pincho Experiments

Laboratory scale Z-pincho showing glow from an expanded hydrogen plasma. Pinch and ionisation current flows through the gas and returns via the bars surrounding the plasma vessel.

(Sandpiper/Wikipedia)
Magnetohydrostatics

- Last time:

\[ \nabla p = \frac{1}{4\pi} (\nabla \times B) \times B \]

- Pressure-balanced column showed that it could be possible to confine a plasma in a column

Laboratory scale Z-pinch showing glow from an expanded hydrogen plasma. Pinch and ionisation current flows through the gas and returns via the bars surrounding the plasma vessel.

(Sandpiper/Wikipedia)
Is Plasma Column Stable?

- Perturb the plasma column and you can break confinement

- Stabilization can be achieved with z-directed B field

http://universe-review.ca/R13-10-NSeqs08.htm
Force Free Fields

• What fields satisfy

\[(\nabla \times \mathbf{B}) \times \mathbf{B} = 0\]

- \(\mathbf{B}\) must be parallel to \(\nabla \times \mathbf{B}\)

\[\nabla \times \mathbf{B} = \mu \mathbf{B}\]

• Note: \(\mu\) is not completely arbitrary

\[\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mu \mathbf{B}) = 0 \rightarrow \mathbf{B} \cdot \nabla \mu = 0\]

- \(\mu\) cannot vary along a magnetic field line

• Cases:

  - \(\mu = \) constant: \textit{linear force-free field}
  - \(\mu\) varies from line to line: \textit{nonlinear force-free field}
Linear Force-Free Field

- Chandra and Kendall solved this via series expansion
- We’ll simplify
  - Assume cylindrical symmetry
    \[ \nabla \times \mathbf{B} = -\frac{d B_z}{dr} \hat{\theta} + \frac{1}{r} \frac{d}{dr} (r B_\theta) \hat{z} \]
  - This gives:
    \[ -\frac{d B_z}{dr} = \mu B_\theta \]
    \[ \frac{1}{r} \frac{d}{dr} (r B_\theta) = \mu B_z \]
  - Solution (blackboard)
    \[ B_\theta = B_0 J_1 (\mu r) \; ; \; B_z = B_0 J_0 (\mu r) \]
Nonlinear Force-Free Field

- Allow $\mu$ to vary from one field to the next
- Start with our expression from pressure-balanced column:

$$
\frac{d}{dr} \left( p + \frac{B_\theta^2 + B_z^2}{8\pi} \right) + \frac{B_\theta^2}{4\pi r} = 0
$$

- Take $p = 0$ (force-free)
- Write magnetic pressure as generating function $F(r)$

$$
\frac{dF}{dr} + \frac{B_\theta^2}{4\pi r} = 0
$$

- Components:

$$
B_\theta^2 = -4\pi r \frac{dF}{dr}
$$

$$
B_z^2 = 8\pi \left( F(r) + \frac{1}{2} r \frac{dF}{dr} \right)
$$
Nonlinear Force-Free Field

• Requirements to be real:
  – $B_\theta$: $\frac{dF}{dr} < 0$
  – $B_z$:

$$F(r) + \frac{1}{2} r \frac{dF}{dr} \geq 0$$

• Any $F(r)$ that satisfies these constraints gives a non-linear force-free field.
Hydromagnetic Waves

• Previously we did perturbation analysis for the neutral fluid equations and found disturbances propagate at the sound speed.

• Now we want to do the same with the ideal MHD equations

• We’ll follow the approach from Spruit (linked on the website)
  – Choudhuri’s approach is more complex, but we get the same result

• Main result: there are 3 different waves
  – Alfvén waves
  – Slow and fast magnetoacoustic waves
Hydromagnetic Waves

- Take plasma with uniform $\mathbf{B} = B\mathbf{\hat{z}}$
- Fluid is uniform and at rest
- We can make the problem two-dimensional
  - $\mathbf{B}$ field direction is preferred
  - 2 orthogonal directions are the same
- We’ll make the perturbations independent of $y$

\[
\begin{align*}
\rho & \rightarrow \rho + \delta\rho(x, z, t) \\
p & \rightarrow p + \delta p(x, z, t) \\
\mathbf{v} & \rightarrow 0 + \mathbf{v}(x, z, t) \\
\mathbf{B} & \rightarrow \mathbf{B} + \delta\mathbf{B}(x, z, t)
\end{align*}
\]
Hydromagnetic Waves

- Basic approach
  - Find perturbed equations to first order
  - Introduce plane wave form of solution
  - Require determinant of system be zero (for non-trivial solutions)

- We’ll work this out on the blackboard

- New velocity scale, Alfvén speed:

\[ v_A = \frac{B}{\sqrt{4\pi \rho}} \]
Hydromagnetic Waves

- Take angle between propagation direction, $\mathbf{k}$, and $\mathbf{B}$ as $\theta$
- Waves:
  - Alfvén wave:
    $$v = \frac{\omega}{k} = v_A \cos \theta$$
  - Slow/fast magnetoacoustic:
    $$v^2 = \frac{1}{2} \left( c_s^2 + v_A^2 \right) \pm \frac{1}{2} \sqrt{\left( c_s^2 + v_A^2 \right)^2 - 4 c_s^2 v_A^2 \cos \theta}$$
Hydromagnetic Waves

- Alfvén waves propagate along field line and are:
  - Incompressive
  - Transverse ($\delta B$ and $\mathbf{v}$ are perpendicular to field)
  - Non-dispersive

- Slow/fast magnetoacoustic:
  - $\delta \rho$ and $\delta p$ are not zero
  - Phase speed ($\omega / k$) depends on angle between wave vector $\mathbf{k}$ and $\mathbf{B}$
  - Direction of displacement varies from along $\mathbf{B}$ to perpendicular
    (see the review by Spruit for some nice figures)
Magnetohydrodynamics
Sunspots

- Sunspots: regions slightly cooler than their surroundings ($T \sim 3800$ K)
Sunspots

Credit: Vacuum Tower Telescope, NSO, NOAO
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Sunspots

- Zeeman effect (splitting of atomic energy levels in B field) allows us to measure magnetic field strength
- Sunspots also appear in pairs
- Polarity is different in north and south hemispheres
“An unusual type of solar eclipse occurred last year. Usually it is the Earth’s Moon that eclipses the Sun. Last June, most unusually, the planet Venus took a turn. Like a solar eclipse by the Moon, the phase of Venus became a continually thinner crescent as Venus became increasingly better aligned with the Sun. Eventually the alignment became perfect and the phase of Venus dropped to zero. The dark spot of Venus crossed our parent star. The situation could technically be labeled a Venusian annular eclipse with an extraordinarily large ring of fire. Pictured above during the occultation, the Sun was imaged in three colors of ultraviolet light by the Earth-orbiting Solar Dynamics Observatory, with the dark region toward the right corresponding to a coronal hole. Hours later, as Venus continued in its orbit, a slight crescent phase appeared again. The next Venusian solar eclipse will occur in 2117.”

http://apod.nasa.gov/apod/ap130820.html
Magnetic Buoyancy

- Convection bundles up magnetic fields into flux tubes
- Sunspots occur in pairs with opposite polarity
  - Magnetic flux tube breaks out through the surface
  - Sunspots mark the location where the flux tube exits and enters the surface
- Pressure balance

\[
P_{\text{out}} = P_{\text{in}} + \frac{B^2}{8\pi}
\]

\[P_{\text{in}} < P_{\text{out}}\]
Magnetic Buoyancy

- Gas pressure difference also implies fluid density inside the flux tube is lower

- Consider constant T
  - Ideal gas law:
    \[ R \rho_{\text{out}} T = R \rho_{\text{in}} T + \frac{B^2}{8\pi} \]

  - Buoyancy:
    \[ \frac{\rho_{\text{out}} - \rho_{\text{in}}}{\rho_{\text{out}}} = \frac{B^2}{8\pi p_{\text{out}}} \]
Magnetic Buoyancy

- High $R_m$: flux is frozen into fluid
- Flux tube is buoyant and rises (*magnetic buoyancy*)
- Sunspots:
  - Middle of a flux tube become buoyant
  - Flux tube breaks through surface
  - Temperature is also lower
    - Magnetic pressure in sunspots “replaces” some of the gas pressure
    - Sunspots appear darker than their surroundings
Sunspots

- Sun exhibits differential rotation
Sunspots

- Flux freezing and differential rotation can explain the presence of sunspots and change in polarity from north to south hemisphere.

(from Bennett et al.)
Parker Instability

- ISM appears clumpy—why?
Parker Instability

- Parker showed that a uniform ISM is unstable
  - Magnetic buoyancy will cause clumps
- Ideal MHD: B field frozen into the ISM
- Consider an initially uniform ISM with parallel B field:
Parker Instability

- Perturb the field
  - Magnetic buoyancy will cause clumps
- Ideal MHD: B field frozen into the ISM
- Plasma cannot fall through lines, so it slides along them
Parker Instability

- Loss of plasma where we are most buoyant makes flux tube more buoyant
- Density enhancements form next to bulging field
- Magnetic tension eventually halts rise
Ferraro’s Law

- Consider axisymmetric rotating object
- Cylindrical coords:
  \[ \mathbf{v} = r\Omega(r, z)\hat{\theta} \]
  - Note: \( \Omega \) is independent of \( \theta \)
- Imagine that there is an axisymmetric *poloidal* B field
  - Ideal MHD: frozen into the plasma
- **Ferraro’s law of isorotation**: steady state only possible if \( \Omega \) is constant along B lines
  - Proof: blackboard
- If \( \Omega \) were to vary along field lines, we’d have differential rotation
  - Poloidal field would turn into toroidal field
Magnetic Braking

- Consider collapse of cloud during star formation
  - Conservation of angular momentum $\rightarrow$ cloud spins up
  - Ideal MHD: B flux frozen into plasma $\rightarrow$ magnetic field strengthens

- Conservation of angular momentum: $\Omega r^2 = \text{const}$ (per unit mass)
  - $\Omega$ of cloud will be much higher than surroundings
  - B field will resist this difference and attempt to slow rotation: *magnetic braking*
Magnetic Braking

Credit: NASA/JPL-Caltech/R. Hurt (SSC).
Magnetic Braking

- Imagine surroundings rotate with cloud angular velocity $\Omega$ out to radius $r = a$
  - $\Omega_{\text{out}} \ll \Omega$ for $r > a$
  - Magnetic stresses will try to spin up plasma outside of $r = a$
  - Magnetic disturbances propagate at Alfvén speed

- Simple estimate of angular momentum transfer (blackboard):

$$\frac{2}{5} M a^2 \frac{d\Omega}{dt} = - \frac{8\pi}{3} a^4 \rho v_A \Omega$$
Magnetized Winds

- Previously we looked at the solar wind
  - But the Sun is rotating and is magnetized
  - Does Ferraro’s law hold?
    - Our derivation needed a velocity of the form:
      \[ \mathbf{v} = r \Omega(r, z) \hat{\theta} \]
    - Now we have a radial component: the solar wind
- Lower corona: magnetic energy density dominates
  \[ \frac{B^2}{8\pi} \gg \frac{1}{2} \rho v^2 \]
  - Magnetic stresses should give rigid rotation
- Further out along solar wind, \( \Omega \) decreases
  - Field lines spiral (*Parker spirals*)
Magnetized Winds

https://ase.tufts.edu/cosmos/

https://ned.ipac.caltech.edu/level5/March03/Vallee2/Vallee3_2.html

PHY 688: Astrophysical Fluids and Plasmas
Magnetized Winds

- Distance out to which magnetic energy dominates over KE: Alfvén radius
- Plasma rotates approximately like solid body out to Alfvén radius
  - No B: angular momentum / unit mass carried by solar wind is $\Omega_\odot R_\odot^2$ (just surface amount)
  - B field makes plasma rotate at angular velocity $\Omega_\odot$ out to $r_A$, where angular momentum / unit mass is $\Omega_\odot r_A^2$
    - This is the angular momentum that can be carried by the solar wind
  - Detailed calculations: $r_A \sim 10 R_\odot$
    - Solar wind is very efficient at carrying angular momentum
Jets

- We looked at the Blanford & Rees model of jets
- What about hydromagnetic jets?
  - This was discussed by Blanford & Payne
- Basic idea:
  - Energy release from accretion onto central object (black hole)
    power jets on polar axes
  - B fields can be used to move gas from disk, carrying angular momentum
  - This ejected gas can be collimated to form a jet
- Blackboard derivation: force along field line is outward if field line makes an angle < 60° with accretion disk
  - This allows plasma to escape from the disk
Jets

- Result of outflow from disk:
  - Magnetic stresses force plasma to rotate with $\Omega$ from where it left disk out to some Alfvén distance
  - Beyond Alfvén distance,
    \[
    \frac{B^2}{8\pi} < \frac{1}{2} \rho u^2
    \]
  - Just like solar wind, angular momentum / unit mass carried by outflow is larger than value on disk
Jets

- Why collimated?
  - Blanford & Payne: beyond Alfvén distance, plasma rotates with lower angular velocity, B field lines twist
  - Magnetic stresses force outflow to be along polar axis

(Marscher et al. 2008)