Low Speed Hydrodynamics
Notes

- In addition to the slides and code examples, my notes on PDEs with the finite-volume method are up online:
  - https://github.com/Open-Astrophysics-Bookshelf/numerical_exercises
Timestep Limitations

- We developed explicit numerical methods for the Euler equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0
\]

\[
\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{U} + \mathbf{p}) = 0
\]

\[
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{E} + \mathbf{p} \mathbf{U}) = 0
\]

- Timestep is restricted such that information can only propagate across one zone per timestep.
- Characteristic speeds are the eigenvalues of the system.
- For the Euler equations, these speeds are:

\[\lambda = u, u \pm c\]
Timestep Limitations

- Stability analysis tells us that the timestep restriction for an explicit scheme solving these equations is:

\[
\Delta t = \min \left\{ \frac{\Delta x}{|u| + c} \right\}
\]

- If the Mach number is small \((M \ll 1)\), then

\[
\Delta t = \min \left\{ \frac{\Delta x}{c} \frac{1}{1 + |M|} \right\} \approx \min \left\{ \frac{\Delta x}{c} (1 - |M|) \right\}
\]

- For \(M \to 0\), we have

\[
\Delta t \approx \frac{\Delta x}{c}
\]
Timestep Limitations

- For very low Mach number flows, it will take $\sim 1/M$ steps for a fluid element to move more than one zone in the simulation.

- We want a method that allows timesteps based on the bulk fluid velocity instead of the sound speed.
  - Previously, we developed the von Neumann stability analysis to tell us how large our timesteps could be.
  - For the linear advection equation,
    \[ a_t + ua_x = 0 \]
    we saw that the explicit discretization,
    \[ a_{i}^{n+1} = a_{i}^{n} - C(a_{i}^{n} - a_{i-1}^{n}) \]
    is stable for
    \[ C = (\Delta tu)/\Delta x < 1 \]
Implicit Differencing

- Alternately, we can use an implicit discretization:

\[ a_i^{n+1} = a_i^n - C(a_i^{n+1} - a_{i-1}^{n+1}) \]

  - This is stable for any value of C

- The differencing above is only first order accurate
  - It is possible to construct higher-order accurate implicit methods as well.
Implicit Differencing

- Implicit methods require solving a linear system.
  - For the 3-d Euler equations, there are 5 unknowns (\(\rho, \rho u, \rho v, \rho w, \rho E\)) on a grid with \(N^3\) zones.
  - Matrix is \(5 \times 5 \times N^3\)—this is very large

- Even accounting for the fact that the matrix is sparse, fully implicit methods can be quite computationally demanding.

- Instead, we can reformulate the equations to identify those terms which model the compressible effects and treat them implicitly and perform the rest of the update explicitly
  - This is the approach taken by low speed formulations
Low Mach Hydrodynamics

• With explicit time-stepping, information cannot propagate more than one zone per step

\[ \Delta t = \min \left\{ \frac{\Delta x}{|u| + c} \right\} \]

• For \( M \ll 1 \):

\[ \Delta t \approx \frac{\Delta x}{c} \]

• We want:

\[ \Delta t \approx \frac{\Delta x}{|u|} \]

• For very low Mach number flows, it takes \( \sim 1/M \) timesteps for a fluid element to move more than one zone—can't we do better?

A Mach 0.01 front moving to the right:

(a) initially, (b) after 1 step, (c) after 100 steps.
Incompressible Flow

- The simplest low Mach number equation set are the incompressible equations for a constant density fluid.
  - Assumes the density of a fluid parcel does not change.
    \[
    \frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{U} = 0
    \]
  - Introduces a constraint on the velocity field
    \[
    \nabla \cdot \mathbf{U} = 0
    \]
  - Compliments the momentum equation:
    \[
    \mathbf{U}_t + (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla p = 0
    \]
- No need for an equation of state (2 equations + two unknowns)
  - The role of the pressure is to enforce the divergence constraint.
Incompressible Flow

- Incompressible equations are formally the zero Mach number limit of the Euler equations.
  - Start with momentum equation:
    \[
    \frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{U}) + \nabla p = 0
    \]

- Seek the Mach number dependence of each term by performing a scaling:
  \[
  \bar{U} = \frac{U}{|U_0|}; \quad \bar{t} = \frac{t}{t_0}; \quad \bar{\rho} = \frac{\rho}{\rho_0}; \quad \bar{x} = \frac{x}{L_0}; \quad \bar{\rho} = \frac{\rho}{\rho_0 c_0^2}
  \]

- Defining Mach number, \( M = |U_0|/c_0 \),
  \[
  \frac{\partial (\bar{\rho} \bar{U})}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\rho} \bar{U} \bar{U}) + \frac{1}{M^2} \bar{\nabla} \bar{p} = 0
  \]
Incompressible Flow

• Expanding the pressure in terms of Mach number,

\[ p = p_0 + p_1 M + p_2 M^2 \]

– Introducing this into the momentum equation, we see

\[ \nabla p_0 = \nabla p_1 = 0 \]

\[ \frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U U) + \nabla p_2 = 0 \]

• We can combine \( p_0 \) and \( p_1 \)

– In an open domain, they will be constant

• Only the *dynamical pressure* appears in the momentum equation.
Incompressible Flow

- Recall that in the primitive variable formulation, the pressure equation is:

\[
\frac{\partial p}{\partial t} + \gamma p \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla p = 0
\]

- General divergence constraint is

\[
\nabla \cdot \mathbf{U} = -\frac{1}{\gamma p} \frac{Dp}{Dt}
\]

- Taking our pressure expansion:

\[
p(x, t) = p_0 + M^2 p_2(x, t)
\]

- We see that

\[
\nabla \cdot \mathbf{U} \sim O(M^2)
\]

- Incompressible system is valid at low Mach numbers
Incompressible Flow

- The incompressible system is not hyperbolic:

\[ U_t + (U \cdot \nabla)U = -\nabla p \]
\[ \nabla \cdot U = 0 \]

- Divergence constraint implies infinitely fast acoustic equilibration

- Different solution techniques are required for this system
  - A powerful technique for solving these equations is the *projection method* developed by Chorin (1968)
Methods for Incompressible Flow

- Any vector field $\mathbf{U}$ can be uniquely decomposed into a divergence free part and the gradient of a scalar:
  \[ \mathbf{U} = \mathbf{U}^d + \nabla \phi \]
  - Subjected to the boundary condition (at walls):
    \[ \mathbf{U}^d \cdot \hat{n} = 0 \]
  - This is called a Hodge decomposition

- Consider the inner product:
  \[ \int_{\Omega} \mathbf{U}^d \cdot \nabla \phi \, d\Omega = \int_{\Omega} \nabla \cdot (\mathbf{U}^d \phi) \, d\Omega - \int_{\Omega} \phi \nabla \cdot \mathbf{U}^d \, d\Omega \]
  \[ = \int_{\partial \Omega} \phi \mathbf{U}^d \cdot \hat{n} \, dS = 0 \]
  - Other boundary conditions can be accommodated with slight variations.
Methods for Incompressible Flow

- We can define a projection operator to project out the divergence free component of a vector:

\[ P = I - \nabla (\Delta)^{-1} \nabla. \]

- Clearly:

\[ PU = U^d; \quad (1 - P)U = \nabla \phi \]

- We can write our incompressible momentum equation as:

\[ U_t + \nabla p = -(U \cdot \nabla)U \]

- On the left hand side, we see a divergence free term plus the gradient of a scalar.

- Applying the projection operator:

\[ U_t = P \left( -(U \cdot \nabla)U \right) \]

- By definition, this satisfies the divergence constraint.
Methods for Incompressible Flow

- Solving this system consists of advection and projection steps.
  - Advection step: a provisional velocity field is predicted using a 2nd order accurate Godunov method:
    \[ \mathbf{U}^* = \mathbf{U}^n - \Delta t \left[ (\mathbf{U}^{ADV} \cdot \nabla) \mathbf{U} \right]^{n+1/2} - \Delta t \nabla p^{n-1/2} \]
    - \( \mathbf{U}^{ADV} \) is constructed to satisfy the divergence constraint at the \( \frac{1}{2} \) time.
    - \( \mathbf{U}^* \) does not satisfy the divergence constraint.
  - Projection step: project this provisional velocity onto the space of discretely divergence free vectors, by solving an elliptic equation:
    \[ \nabla^2 \phi = \nabla \cdot \mathbf{U}^* \]
    - Final velocity is then found via:
      \[ \mathbf{U}^{n+1} = \mathbf{U}^* - \nabla \phi \]
Methods for Incompressible Flow

- These methods are more complicated to implement than the compressible methods we studied up to now.
- Benefit is that we can take timesteps that are constrained by

\[ \Delta t = \min \left\{ \frac{\Delta x}{|U|} \right\} \]

A doubly periodic shear layer (vorticity shown) computed with 64 zones (left), 128 zones (center), and 256 zones (right) across.
Low Mach Number Combustion

- Incompressible flow does not allow for any compressibility
  - There are less restrictive equation sets that still filter sound waves.

- Consider a reacting fluid
  - In addition to the normal evolution equations, we carry an advection equation for each chemical/nuclear species:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0
\]

\[
\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{U} + p) = \rho \mathbf{g}
\]

\[
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot [(\rho E + p) \mathbf{U}] = \nabla \cdot (\kappa \nabla T) + \rho \mathbf{U} \cdot \mathbf{g} + \dot{\mathcal{H}}
\]

\[
\frac{\partial (\rho X_k)}{\partial t} + \nabla \cdot (\rho \mathbf{U} X_k) = \rho \dot{\omega}_k
\]
Low Mach Number Combustion

- Continuity equation is redundant, since the species variables are defined such that:

\[ X_k = \frac{\rho_k}{\rho}; \quad \sum_k X_k = 1 \]

- Consider the enthalpy evolution instead of total energy:

\[ \rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = \nabla \cdot \kappa \nabla T + \dot{H} \]

- We can non-dimensionalize this system, and write the pressure in terms of Mach number:

\[ p(x, t) = p_0(t) + M p_1(t) + M^2 \pi(x, t) \]
Low Mach Number Combustion

- As with incompressible set, only the dynamic pressure appears in the momentum equation:

\[
\frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{U}) = -\nabla \pi + \rho \mathbf{g}
\]

- Open domain: pressure can be assumed constant, so

\[
\frac{Dp_0}{Dt} = 0
\]

- Replacing \( p \) by \( p_0 \) in our enthalpy equation, we have:

\[
\rho \frac{Dh}{Dt} = \nabla \cdot \kappa \nabla T + \dot{H}
\]

- Note: we have not used the equation of state.
  - All the thermodynamics of the system is all contained in \( p_0 \)
Low Mach Number Combustion

- Use the equation of state to derive a constraint on the velocity:

\[
\frac{Dp_0}{Dt} = 0 = \frac{\partial p}{\partial \rho} \frac{D\rho}{Dt} + \frac{\partial p}{\partial T} \frac{DT}{Dt} + \sum_k \frac{\partial p}{\partial X_k} \frac{DX_k}{Dt}
\]

- Mass continuity:

\[
\nabla \cdot \mathbf{U} = \frac{1}{\rho} \frac{\partial \rho}{\partial T} \left( \frac{\partial p}{\partial T} \frac{DT}{Dt} + \sum_k \frac{\partial p}{\partial X_k} \frac{DX_k}{Dt} \right)
\]

- We need a temperature evolution equation:

\[
\frac{Dh}{Dt} = \frac{\partial h}{\partial T} \bigg|_{p, X_k} \frac{DT}{Dt} + \frac{\partial h}{\partial p} \bigg|_{T, X_k} \frac{Dp}{Dt} + \sum_k \frac{\partial h}{\partial X_k} \bigg|_{p, T, X_{j, j \neq k}} \frac{DX_k}{Dt} + c_p
\]
Low Mach Number Combustion

- This yields:
  \[ \rho c_p \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + \dot{H} - \sum_k \rho \frac{\partial h}{\partial X_k} \dot{\omega}_k \]

  - Our divergence constraint becomes:
    \[ \nabla \cdot \mathbf{U} = S \]
    \[ S \equiv \frac{1}{\rho} \frac{\partial p}{\partial \rho} \left( \frac{1}{\rho c_p} \frac{\partial p}{\partial T} \left( \nabla \cdot (k \nabla T) + \dot{H} - \sum_k \rho \frac{\partial h}{\partial X_k} \dot{\omega}_k \right) + \sum_k \frac{\partial p}{\partial X_k} \dot{\omega}_k \right) \]

- This looks like the incompressible constraint, but with a source
  - Captures compressibility effects due to thermal diffusion and reactions

- This formulation links two incompressible states (fuel and ash) by the expansion at the reaction front.
As $\rho$ decreases, RT dominates over burning. At low $\rho$, flame width is set by mixing scale.

PHY 688: Astrophysical Fluids and Plasmas
We saw to model low speed flows and include the compressibility effects due to diffusion and reactions.

- These models are valid when we are interested in the dynamics on scales much less than a pressure scale height.

For largescale stratification, we need to incorporate additionally compressibility effects.

- To zeroth order, the system is in hydrostatic equilibrium:

$$\nabla p_0 = \rho_0 \mathbf{g}$$

Asymptotic analysis is more complicated (see Klein & Pauluis 2012), but results in:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + \frac{\beta_0}{\rho} \left( \frac{p'}{\beta_0} \right) = \frac{\rho'}{\rho} \mathbf{g}$$

- Here, $\beta_0 = p_0^\gamma$ (for ideal gas)
• Constraint is developed in much the same way as the low Mach number combustion system, except now, $\frac{Dp_0}{Dt} \neq 0$

  - We get:

\[
\nabla \cdot U + \frac{1}{\Gamma_1 p_0} \left( \frac{\partial p_0}{\partial t} + U \cdot \nabla p_0 \right) = S
\]

  - Or compactly:

\[
\nabla \cdot (\beta_0 U) = \beta_0 S
\]

  - Neglecting time-dependence in $p_0$

• This low Mach number formulation includes the compressibility effects due to both the background stratification and the local energy sources.

  - It is possible to self-consistently evolve the background state (which is in hydrostatic equilibrium) due to the energy release.
Here we see the Mach number of the flow resulting from a temperature perturbation in a stratified atmosphere. The fully compressible solution on the left shows a pressure wave propagating outward from the perturbation, communicating the disturbance at a finite speed. For the low Mach number results on the right, the communication was instantaneous.
Popular Low Speed Approximations

- **Incompressible**
  - No compressibility effects modeled

- **Anelastic**
  - Compressibility due to stratified atmosphere
  - Small thermodynamic perturbations from a static hydrostatic (usually isentropic) background

- **Low Mach number combustion**
  - Local compressibility due to heat release and diffusion
  - Large temperature/density jumps permitted

- **Pseudo-incompressible**
  - Both compressibility due to background stratification and local heat release
  - Static background
  - Originally: ideal gas EOS

\[ \nabla \cdot U = 0 \]
\[ \nabla \cdot (\rho_0 U) = 0 \]
\[ \nabla \cdot U = S \]
\[ \nabla \cdot (p_0^{1/\gamma} U) = S \]
**Incompressible**: density of a fluid element doesn’t change as it is advected

\[
\nabla \cdot U = 0
\]

\[
- \frac{1}{\rho} \frac{D\rho}{Dt}
\]

**Low Mach combustion**: heat release in fluid element is a source of divergence, fluid element expands

\[
\nabla \cdot (\rho U) = 0
\]

**Anelastic**: fluid element adiabatically expands as it buoyantly rises