Krook approximation (follow Shu problem sets)

We'll write the Krook equation as

\[ \mathcal{L} f = - \mathcal{V}_c (f - f_0) \]

\[ \mathcal{L} \text{ is Maxwell-Boltzmann Boltzmann operator} \]

Using a series expansion, \( f = f_0 + f_1 + \ldots \)

0th order solution is \( f = f_0 \)

Next order:

\[ \mathcal{L} (f_0 + f_1) = - \mathcal{V}_c (f_0 + f_1 - f_0) = - \mathcal{V}_c f_1 \]

Keeping only lowest order term on left,

\[ f_1 \sim - \frac{1}{\mathcal{V}_c} \mathcal{L} f_0 \]

by construction \( f_1 \ll f_0 \)

Recall that

\[ \Pi_{ik} = \rho \left< \frac{i}{2} \hat{w} \hat{w}^* S_{ik} - w_i w_k \right> \]

\[ F_i = \rho \left< w_i \frac{1}{2} |\hat{w}|^2 \right> \]
What is $\pi_{ik}$?

To 1st order,

$$\pi_{ik} = m \int \left( \frac{1}{2} \hat{\mathbf{w}} \cdot \mathbf{W}_{ik} - \mathbf{w} \cdot \mathbf{w} \right) \left( f_0 + f_1 \right) \, d^3 \mathbf{w}$$

$$\text{where } \mathbf{p} = \frac{\mathbf{m} \mathbf{w}}{\sqrt{m}}$$

we already did the integral w/ $f_0$, and that's how we get the $p$ term, so it does not contribute here (since $f_0$ is isotropic, and off-diagonal terms vanish)

That leaves

$$\pi_{ik} = m \int \left( \frac{1}{2} \hat{\mathbf{w}} \cdot \mathbf{W}_{ik} \right) f_1 \, d^3 \mathbf{w}$$

$$= -\frac{m}{\mathcal{V}_c} \int \left( \frac{1}{2} \hat{\mathbf{w}} \cdot \mathbf{W}_{ik} \right) \mathcal{L} f_0 \, d^2 \mathbf{w}$$

(note: depending on the source, there is a sign difference overall)

Also,

$$F_i = m \int \frac{1}{2} \mathbf{w} \cdot |\hat{\mathbf{w}}|^2 \left( f_0 + f_1 \right)$$

we did the $f_0$ integral previously and it was 0, so that leaves just $f_1$

$$\therefore F_i = m \int \frac{1}{2} \mathbf{w} \cdot |\hat{\mathbf{w}}|^2 f_1 = -\frac{m}{\mathcal{V}_c} \int \frac{1}{2} |\hat{\mathbf{w}}|^2 \mathbf{w} \cdot \mathcal{L} f_0 \, d^3 \mathbf{w}$$
We need to transform between \((\hat{x}, \hat{v}, t)\) and \((\hat{x}, \hat{w}, t)\), keeping in mind that \(\hat{w} = \hat{v} - \hat{u}(x, t)\) (by definition).

Consider a function \(g(\hat{x}, \hat{w}, t)\), we can write this as

\[ g = g(\hat{x}, \hat{w}(\hat{x}, \hat{v}, t), t) \]

Now differentiate:

1. derivative of \(t\) with \(\hat{x}, \hat{v}\) fixed

\[
\frac{dg}{dt}|_{\hat{x}, \hat{v}} = \frac{dg}{dt}|_{\hat{x}, \hat{w}} + \frac{dg}{d\hat{w}}|_{\hat{x}, t} \frac{d\hat{w}}{dt}|_{\hat{x}, \hat{v}}
\]

Now \(w = v - u(x, t)\), so

\[
\frac{d\hat{w}}{dt}|_{\hat{x}, \hat{v}} = -\frac{du}{dt}
\]

and we have

\[
\left( \frac{\partial}{\partial t} \right)|_{\hat{x}, \hat{v}} = \left( \frac{\partial}{\partial \hat{w}} \right)|_{\hat{x}, \hat{w}} - \frac{du}{dt} \frac{\partial g}{\partial \hat{w}}|_{\hat{x}, t}
\]

2. differentiate \(w\) wrt \(x\), keeping \(t, v\) fixed

\[
\frac{dg}{dx}|_{\hat{v}, t} = \frac{dg}{dx}|_{\hat{w}, t} + \frac{dg}{d\hat{w}}|_{\hat{x}, t} \frac{d\hat{w}}{dx}|_{\hat{x}, \hat{v}, t}
\]

but \(\frac{d\hat{w}}{dx}|_{\hat{x}, \hat{v}, t} = -\frac{dv}{dx}\) (from \(w = v - u(x, t)\))

\[
\therefore \left( \frac{\partial}{\partial x} \right)|_{\hat{v}, t} = \left( \frac{\partial}{\partial \hat{w}} \right)|_{\hat{w}, t} - \frac{du}{dx} \frac{\partial g}{\partial \hat{w}}|_{\hat{x}, t}
\]
finally, differentiating w.r.t. \( v \) with \( x, t \) fixed

\[
\frac{\partial g}{\partial v_i} \bigg|_{x,t} = \frac{\partial g}{\partial w_j} \frac{\partial w_j}{\partial v_i} \bigg|_{x,t}
\]

but \( w_j = v_j - u_j(x,t) \)

so \( \frac{\partial w_j}{\partial v_i} = \delta_{ij} \)

\[
\therefore \left( \frac{\partial}{\partial v_i} \right)_{x,t} = \left( \frac{\partial}{\partial w_i} \right)_{x,t}
\]

This expression matches

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Now

\[
L f_0 = \frac{\partial f_0}{\partial t} \bigg|_{x,v} + v_i \frac{\partial f_0}{\partial x_i} \bigg|_{v,t} + \frac{F_i}{m} \frac{\partial f_0}{\partial v_i} \bigg|_{x,t}
\]

using our transformed derivatives

\[
L f_0 = \frac{\partial f_0}{\partial t} \bigg|_{x,w} - \frac{\partial v_j}{\partial t} \frac{\partial f_0}{\partial w_j} \bigg|_{x,t}
\]

\[
+ v_i \left[ \frac{\partial f_0}{\partial x_i} \bigg|_{w,t} - \frac{\partial v_j}{\partial x_i} \frac{\partial f_0}{\partial w_j} \bigg|_{x,t} \right]
\]

\[
+ \frac{F_i}{m} \left[ \frac{\partial f_0}{\partial w_i} \right]_{x,t}
\]
grouping terms:

\[ L f_0 = \frac{\partial f_0}{\partial t} \bigg|_{x,w} + (u_i + w_i) \frac{\partial f_0}{\partial x_i} \bigg|_{w,t} \]

\[ - \left[ \frac{\partial u_j}{\partial t} + (u_i + w_i) \frac{\partial u_j}{\partial x_i} - \frac{F_i}{m} \right] \frac{\partial f_0}{\partial w_j} \bigg|_{x,t} \]

Note that \( f_0(\vec{w}) \) is the Maxwell-Boltzmann distribution

\[ f_0(\vec{w}) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} e^{-m|\vec{w}|^2/2k_B T} \]

Define \( G_{ik}(\vec{w}) = \left( \frac{1}{8} |\vec{w}|^2 S_{ik} - w_i w_k \right) \left(-\frac{m}{\bar{v}_e} \right) \)

so \( \Pi_{ik} = \int G_{ik}(\vec{w}) L f_0(\vec{w}) d^3w \)

we know from our \( \delta \)th order approximation that

\[ \int G_{ik}(\vec{w}) f_0(\vec{w}) d^3w = 0 \]

so we want to see how these terms simplify using this.

we see

\[ \int G_{ik}(\vec{w}) \frac{\partial f_0}{\partial t} \bigg|_{x,w} d^3w = \frac{2}{m} \int G_{ik}(\vec{w}) f_0 d^3w = 0 \]

(\text{note what is constant})

also,

\[ \int G_{ik} u_j \frac{\partial f_0}{\partial x_j} \bigg|_{w,t} d^3w = u_j \frac{\partial}{\partial x_j} \int G_{ik}(\vec{w}) f_0 d^3w = 0 \]

\[ \int G_{ik} w_j \frac{\partial f_0}{\partial x_j} \bigg|_{w,t} d^3w = \frac{\partial}{\partial x_j} \int G_{ik}(\vec{w}) w_j f_0 d^3w = 0 \]

this is because the integrand is an odd function of \( w \).
This then leaves just the last term

\[- \left[ \frac{\partial u_j}{\partial t} + (v_i + w_i) \frac{\partial u_j}{\partial x_i} - \frac{F_i}{m} \right] \frac{\partial f_0}{\partial w_j} \bigg|_{x,t} \]

we can split this into a term proportional to \( w_i \) and the rest

\[= - \left( C_i + w_i \frac{\partial u_j}{\partial x_i} \right) \frac{\partial f_0}{\partial w_j} \bigg|_{x,t} \]

and since this is all that is left of \( \Delta f_0 \), we can write

\[\Pi_{ik} = \ldots - \int \left[ C_i + w_i \frac{\partial u_j}{\partial x_i} \right] \frac{\partial f_0}{\partial w_j} G_{ik}(w) d^3w \]

\[= - C_i \int \frac{\partial f_0}{\partial w_j} G_{ik}(w) d^3w - \frac{\partial u_j}{\partial x_i} \int w_i \frac{\partial f_0}{\partial w_j} G_{ik}(w) d^3w \]

For the first term, we integrate by parts

\[- C_i \int \frac{\partial f_0}{\partial w_j} G_{ik}(w) d^3w = - C_i \left( \frac{f_0}{\Delta} G_{ik}(w) \right)_{x \to \infty} + C_i \int \frac{\partial G_{ik}}{\partial w_j} f_0 d^3w \]

the first term vanishes, since \( f_0 \to 0 \) at \( \infty \)

For the second term, we need

\[\frac{\partial G_{ik}}{\partial w_j} = \frac{\partial}{\partial w_j} \left[ \left( \frac{1}{2} w_j w_k S_{ik} - w_i w_k \right) (-\frac{m}{c^2}) \right] \]

\[= \left( \frac{2}{3} w_j S_{ik} - w_i S_{ik} - w_k S_{ji} \right) (-\frac{m}{c^2}) \]

each of these terms is odd, and \( f_0 \) is isotropic, so

this vanishes \( <w_i> = 0 \)
Finally, we are left with

\[ \pi_{ik} = - \frac{2u_j}{\partial x_i} \int w_d \frac{2f_0}{\partial w_j} G_{ik}(\hat{w}) \, d^3 w \]

\[ = \frac{m}{v_0} \frac{2u_j}{\partial x_i} \int w_d \left( \frac{1}{8} |\hat{w}|^2 \delta_{ik} - w_i w_k \right) \frac{2f_0}{\partial w_j} \, d^3 w \]

Integrating by parts:

\[ = \frac{m}{v_0} \frac{2u_j}{\partial x_i} \left[ \int w_d \left( \frac{1}{8} |\hat{w}|^2 \delta_{ik} - w_i w_k \right) \right]_{\hat{w}} \]

\[ - \int f_0 \frac{2}{\partial w_j} \left[ w_d \left( \frac{1}{8} |\hat{w}|^2 \delta_{ik} - w_i w_k \right) \right] \, d^3 w \]

Computing the derivative in the integrand:

\[ \frac{2}{\partial w_j} \left[ w_d \left( \frac{1}{8} |\hat{w}|^2 \delta_{ik} - w_i w_k \right) \right] \]

\[ = \delta_{jd} \left( \frac{1}{8} |\hat{w}|^2 \delta_{ik} - w_i w_k \right) \]

\[ + w_d \left[ \frac{2}{3} w_i \delta_{ik} - w_i \delta_{jk} - w_k \delta_{ij} \right] \]

Recall that

\[ \int d^5 w_i w_j f(w) = \frac{p}{m} \delta_{ij} \]

(Note that

\[ \int d^3 w_i w_j f(w) = \frac{p}{m} \]

This is not summation)

\[ \int d^3 w_i w_i |\hat{w}|^2 f(w) = \frac{p}{m} \]
Using our previous pressure integral result,

$$\Pi_{ik} = -\frac{m}{v_c} \frac{\partial u_i}{\partial x_k} \left\{ S_{ij} \delta_{ik} \frac{1}{2} \int |W|^2 f_0 \, d^3w \right. \\
- S_{ij} \int W_i W_k f_0 \, d^3w \\
+ \frac{2}{3} S_{ik} \int W_i W_j f_0 \, d^3w \\
- S_{ik} \int W_i W_i f_0 \, d^3w \\
- S_{ij} \int W_i W_k f_0 \, d^3w \left. \right\}$$

$$\Pi_{ik} = \frac{m}{v_c} \frac{\partial u_j}{\partial x_i} \left\{ S_{ik} \delta_{ij} \frac{P}{m} - S_{ij} \delta_{ik} \frac{P}{m} + \frac{2}{3} S_{ik} \delta_{ij} \frac{P}{m} \\
- S_{ik} \delta_{ji} \frac{P}{m} - S_{ij} \delta_{ik} \frac{P}{m} \right\}$$

$$= \frac{n k T}{v_c} \left\{ \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} + \frac{2}{3} \delta_{ik} \frac{\partial u_j}{\partial x_j} \\
- \frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k} \right\}$$

$$\Pi_{ik} = \frac{n k T}{v_c} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial u_j}{\partial x_j} \right) \star$$