Rotation

Many astrophysical systems exhibit differential rotation

In terrestrial system (bucket of water spun around vertical axis) rotation begins differentially (water near walls moves first) but shear means viscous forces act and viscosity leads to the entire bucket spinning with same angular velocity (solid body)

Also:
- Viscous force not strong enough to make system solid body
- Physical mechanism may exist that maintains differential rotation

Steady axisymmetric rotation

\[ \frac{\partial^2}{\partial t} = 0 \quad \frac{\partial}{\partial \theta} = 0 \quad v_r = 0 \]

Navier-Stokes:

\[ \frac{v^2}{r} = g v - \frac{1}{\rho} \frac{\partial p}{\partial r} \] (in cylindrical coordinates)

Slowly rotating star, \( \frac{\partial p}{\partial r} \) and \( pg \) nearly balance

Little part of \( p \) balances centrifugal

We expect these stars to eventually rotate as solid bodies (viscosity)

Molecular viscosity is negligible, but convection can drive turbulence, and turbulent diffusion may allow for differential rotation
Not all types of differential rotation are stable.

Consider a uniform density fluid rotating differentially around an axis of symmetry.

Condition of stability

- Consider fluid ring at distance $r_0$ from axis, moving with velocity $v_0$

Now interchange this ring with a fluid ring at a greater distance $r_1$ ($r_1 > r_0$) moving with velocity $v_1$.

System is stable if displaced fluid rings want to return to their initial positions.

System is unstable if rings want to move further apart.

Conservation of angular momentum:

A ring displaced from $r_0$ to $r_1$ acquires a velocity $(r_0/r_1)v_0$.

A ring previously at $r_1$ had centripetal acceleration $v_1^2/r_1$, which needed to be provided by the pressure in excess of HSE.

The new ring at this position needs centripetal acceleration $r_0^2v_0^2/r_1^3$ to remain in its new position. If this is less than $v_1^2/r_1$, we expect the forces to push it back to its initial position.

Stability: $\frac{r_0^2v_0^2}{r_1^3} < \frac{v_1^2}{r_1}$
conservation of angular momentum

\[ L = v_0 r_0 = v r_1 \]

\[ v = \frac{r_0}{r_1} v_0 \]

new centripetal accel

\[ \frac{r_0^2 v_0^2}{r_1^3} \]

needs to be \( \frac{v_1^2}{r_1} \)
Angular velocity \( v = \frac{2\pi r}{p} = \Omega r \)

so

\[ \frac{r_0^2 \Omega_0^2 r_0^2}{r_1^2 \Omega_1^2 r_1^2} \leq 1 \]

or \( (\Omega_0 r_0^2)^2 < (\Omega_1 r_1^2)^2 \)

This says that \((\Omega_0 r_0^2)^2\) needs to increase w.r.t \( r \), so stability is

\[ \frac{d}{dr} [(r^2\Omega_0^2)] > 0 \]
Rotating reference frame

Imagine sitting in a frame rotating w/ angular velocity $\Omega$

We need to replace acceleration by

$$\frac{dv}{dt} \rightarrow \frac{dv}{dt} + 2\Omega \times v + \dot{\Omega} \times (\Omega \times v)$$

position wrt axis of rotation

derivation (from Castro)

Consider inertial reference frame $C$

and non-inertial frame $C'$

whose origins are separated by vector $l$

Non-inertial frame is rotating about fixed axis w/ angular velocity $\Omega$

Now consider a fluid element at point $P$ whose location

is $r$ in $C$ and $r'$ in $C'$

$$\vec{r} = \vec{r}' + \vec{l}$$

in terms of components

$$r_i e_i = r'_i e'_i + d_i e_i$$

unit vectors on $C$ and $C'$
Total time rate of change is
\[
\frac{D\mathbf{r}_i}{Dt} \mathbf{e}_i = \frac{D\mathbf{r}_i}{Dt} \mathbf{e}_i + r_i \frac{De_i}{Dt} + \frac{D\mathbf{e}_i}{Dt} \mathbf{e}_i
\]

Here, we assume the unit vectors in the inertial frame, C, are not moving.

Unit vectors don't change length, only direction, so the change in \( \mathbf{e}_i \) with time is only due to rotation
\[
\frac{De_i}{Dt} = \Omega \times \mathbf{e}_i \quad \text{(circumferential motion)}
\]

\( \therefore \) we have
\[
\frac{D\mathbf{r}}{Dt} = \frac{D\mathbf{r}'}{Dt} + \Omega \times \mathbf{r}' + \frac{D\mathbf{e}_i}{Dt}
\]

\( \frac{D\mathbf{r}}{Dt} \) velocity of fluid element seen by stationary observer in rotating frame C'
\( \frac{D\mathbf{r}'}{Dt} + \Omega \times \mathbf{r}' + \frac{D\mathbf{e}_i}{Dt} \) additional velocity due to rotation & translation

\( \mathbf{v} \) velocity of fluid element in inertial frame

\( \mathbf{v}' = \mathbf{v} + \Omega \times \mathbf{r}' + v_i \)

Let's assume no translation, so \( v_i = 0 \)

Second derivative:
\[
\frac{2D\dot{\mathbf{v}}}{Dt} = \frac{D^2\mathbf{v}}{Dt}
\]
relation of frames
\[ v_i e_i = v_i' e_i' + \epsilon_{ijk} \Omega_j e_j r_k e_k \]

Second derivative

\[
\frac{Dv_i}{Dt} e_i = \frac{Dv_i'}{Dt} e_i' + v_i' \frac{De_i'}{Dt} \\
+ \epsilon_{ijk} \Omega_j e_j \left[ r_k \frac{De_k'}{Dt} + \frac{Dr_k'}{Dt} e_k' \right]
\]

In inertial frame

\[
= \frac{Dv_i'}{Dt} e_i' + v_i' \Omega \times e_i' + \epsilon_{ijk} \Omega_j e_j r_k \Omega \times e_k' \\
+ \epsilon_{ijk} \Omega_j e_j v_k' e_k' \\
\]  

back to vector form

\[
\frac{D\vec{V}}{Dt} = \frac{D\vec{V}'}{Dt} + \Omega \times \vec{V}' + \Omega \times \left[ \Omega \times \vec{r}' \right] + \Omega \times \vec{V}' \\
= \frac{D\vec{V}'}{Dt} + 2 \Omega \times \vec{V}' + \Omega \times \left[ \Omega \times \vec{r}' \right]
\]
Our momentum $\gamma$ goes from

$$\frac{\partial \gamma}{\partial t} + (\gamma \cdot \nabla)\gamma = -\frac{1}{\rho} \nabla p + f + \nu \nabla^2 \gamma$$

to

$$\frac{\partial \gamma}{\partial t} + (\gamma \cdot \nabla)\gamma = -\frac{1}{\rho} \nabla p + f + \nu \nabla^2 \gamma - 2\Omega \times \gamma - \Omega \times (\Omega \times r)$$

Coriolis centrifugal
Consider
\[ \frac{1}{2} \nabla (|\mathbf{\Omega} \times \mathbf{r}|^2) \]

We'll work in cylindrical coords.
With loss of generality, we can take \( \mathbf{\Omega} \) to be in \( \hat{\mathbf{z}} \) direction, then \( \mathbf{r} \) is in \( \hat{\mathbf{r}} \) direction and is distance from axis.

\[ \mathbf{\Omega} \times \mathbf{r} = \mathbf{\Omega} \hat{\mathbf{z}} \times \mathbf{r} \hat{\mathbf{r}} \]

\[ = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\mathbf{\theta}} & \hat{\mathbf{z}} \\ 0 & 0 & \mathbf{\Omega} \\ r & 0 & 0 \end{vmatrix} \]

\[ = \mathbf{\Omega} r \hat{\mathbf{\theta}} \]

\[ |\mathbf{\Omega} \times \mathbf{r}|^2 = \mathbf{\Omega}^2 r^2 \]

Then \[ \frac{1}{2} \nabla (|\mathbf{\Omega} \times \mathbf{r}|^2) = \frac{1}{2} \frac{\partial}{\partial r} \mathbf{\Omega}^2 r^2 = \mathbf{\Omega}^2 r \]
and

\[ \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \]

\[ = \mathbf{\Omega} \times \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & 0 & \Omega \\ r & 0 & 0 \end{pmatrix} \]

\[ = \mathbf{\Omega} \times (\mathbf{\Omega} \hat{r}) \]

\[ = \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & 0 & \Omega \\ 0 & \Omega r & 0 \end{pmatrix} = -\mathbf{\Omega}^2 \mathbf{r} \]

\[ \therefore -\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \frac{1}{2} \nabla \left( \mathbf{\Omega} \times \mathbf{r} \right)^2 \]
If we write our force as \( f = -\nabla \phi \), then we have

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla p - \nabla \left( \phi - \frac{1}{2} \Omega^2 x r^2 \right) + \nabla^2 v - 2 \Omega x v
\]

this allows us define an effective gravitational potential

\[
\phi_{\text{eff}} = \phi - \frac{1}{2} \Omega^2 x r^2
\]

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When is rotation important?

**Centrifugal**: when a significant part of \( \phi_{\text{eff}} \) is from \(-\frac{1}{2} \Omega^2 x r^2\)

**Coriolis**: compare \((v \cdot \nabla) v\) to Coriolis, \(-2 \Omega x v\)

\[
Ro \equiv \frac{\frac{V^2}{L}}{\Omega V}
\]

Rossby number

Coriolis force is important for \( Ro \lesssim 1 \)
Geostrophic approx

consider that rotation is important, so $\frac{\partial v}{\partial t}$ is small
Also neglect centrifugal and viscosity

$$- \frac{\nabla p}{\rho} - g \hat{e}_r - 2 \Omega \times v = 0$$

vertical direction — Coriolis force is insignificant compared to gravity, so

$$- \frac{\nabla p}{\rho} - g \hat{e}_r = 0$$

or

$$\nabla p = \nabla \rho g$$

gravity is not present in horizontal direction, so

$$\nabla_h p = - 2 \rho (\Omega \times v)_h$$

* indicates lateral

we expect $|\nabla_h p| \ll |\nabla p|$.

A vertical

geostrophic approximation :

$$\frac{1}{\rho} \frac{\partial p}{\partial v} = - g$$

$$\nabla_h p = - 2 \rho (\Omega \times v)_h$$
Consider geometry

Velocity is perpendicular to $\nabla p$ if it supports the pressure gradient.

This shows the flow in a hurricane—swirls over low $p$ region.
Vorticity in rotating frame

Ideal, incompressible fluids

\[ \nabla \cdot \omega = \nabla \cdot \left( \frac{\mathbf{P}}{\rho} \right) \]

and then we write

\[ \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \mathbf{\omega} = -\nabla \left( \frac{\mathbf{P}}{\rho} + \frac{1}{2} \mathbf{v}^2 + \phi - \frac{1}{2} \mathbf{L} \cdot \mathbf{v} + \frac{1}{2} \mathbf{L} \cdot \mathbf{v} \right) - 2\mathbf{\Omega} \times \mathbf{v} \]

\[ \text{Note: we used the vector transform on } (v \cdot \nabla) \mathbf{v} \]

Taking the curl

\[ \frac{\partial \mathbf{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{\omega}) + \nabla \times (\mathbf{v} \times 2\mathbf{\Omega}) \]

We can write this as

\[ \frac{\partial \mathbf{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times (\mathbf{\omega} + 2\mathbf{\Omega})) \]

Note that \( \mathbf{\Omega} \) is constant in time, so we can write

\[ \frac{\partial \mathbf{\omega}}{\partial t} = \frac{\partial}{\partial t} (\mathbf{\omega} + 2\mathbf{\Omega}) \]

and then

\[ \frac{\partial}{\partial t} (\mathbf{\omega} + 2\mathbf{\Omega}) = \nabla \times (\mathbf{v} \times (\mathbf{\omega} + 2\mathbf{\Omega})) \]

This has the form of \( \frac{\partial \mathbf{Q}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{Q}) \) we saw before, so

\[ \frac{d}{dt} \int_{\mathbf{S}} \mathbf{Q} \cdot d\mathbf{S} = 0 \]
\[ \frac{d}{dt} \int_s (\mathbf{\omega} + \mathbf{2} \Omega) \cdot ds = 0 \]

This is a generalization of Kelvin's vorticity theorem for frames that rotate.

aka Bjercknes's theorem
Taylor–Proodman theorem

Consider steady fluid flows in rotating frame

\[
\frac{\partial}{\partial t} (\mathbf{v} + 2\mathbf{\Omega}) = \nabla \times [\mathbf{v} \times (\mathbf{v} + 2\mathbf{\Omega})]
\]

\[
\Rightarrow \nabla \times [\mathbf{v} \times (\mathbf{v} + 2\mathbf{\Omega})] = 0
\]

Assume slow flow, so vorticity is small compared to \(2\mathbf{\Omega}\)

then

\[
\nabla \times (\mathbf{v} \times 2\mathbf{\Omega}) = 0
\]

or

\[
\nabla \times (A \times B) = A (\nabla \cdot B) - B (\nabla \cdot A) + (B \cdot \nabla) A - (A \cdot \nabla) B
\]

\[
\nabla \times (\mathbf{v} \times 2\mathbf{\Omega}) = \mathbf{v} (\nabla \cdot 2\mathbf{\Omega}) - 2\mathbf{\Omega} (\nabla \cdot \mathbf{v}) + (\mathbf{\Omega} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) 2\mathbf{\Omega}
\]

\[
\mathbf{\Omega} = \text{const} \quad \text{incompressible} \quad \mathbf{v} = \text{const}
\]

\[
\Rightarrow (2\mathbf{\Omega} \cdot \nabla) \mathbf{v} = 0
\]

This says that \(\mathbf{v}\) does not change in the direction of \(2\mathbf{\Omega}\)

slow steady flows in rotating frames tend to be invariant parallel to rotation axis

— Taylor–Proodman theorem