Consider a dilute neutral gas

\[ \text{volume of gas particles} \ll \text{volume of container} \]

\[ V = \text{volume of container} \]
\[ a^3 = \text{volume of particle} \]

We need \( Na^3 \ll V \implies na^3 \ll 1 \)

Interactions only when particles collide (separation \( \sim 2a \))
- no long range forces

We already discussed mean free path and saw

\[ \lambda \sim \frac{1}{n_0} \quad \text{(your text has a } \frac{1}{\sqrt{n}} \text{ in this)} \]

Particles move in straight line between collisions

Note: \( \lambda \sim \frac{1}{na^2} \implies na^2 \sim \frac{1}{\lambda} \)

Multiply by \( a \)
\[ na^3 \sim \frac{a}{\lambda} \ll 1 \]

\[ \therefore \lambda \gg a \]
We will neglect three body interactions — diluteness makes this unlikely

Collisionless Boltzmann
\[
\frac{Df}{Dt} = 0
\]

Collisions add a source:

- Particles having velocity \( \mathbf{v} \) initially have a different velocity after collision — \( f(\mathbf{x}, \mathbf{v}, t) \) decreases
- Particles having different velocity may get velocity \( \mathbf{v} \) after collision — \( f(\mathbf{x}, \mathbf{v}, t) \) increases

W/ collisions:
\[
\frac{Df}{Dt} d^3x d^3v = -C_{out} + C_{in}
\]

rates at which particles leave/enter volume \( d^3x d^3v \) of \( \mu \)-space from collisions

Binary collisions
Initial particle velocities: \( \mathbf{v}, \mathbf{v}', \)
After collision: \( \mathbf{v}', \mathbf{v}_i' \)

Conservation of momentum & energy
\[
\mathbf{v} + \mathbf{v}_1 = \mathbf{v}' + \mathbf{v}_i'
\]
\[
\frac{1}{2} |\mathbf{v}|^2 + \frac{1}{2} |\mathbf{v}_i|^2 = \frac{1}{2} |\mathbf{v}'|^2 + \frac{1}{2} |\mathbf{v}_i'|^2
\]
We want $\dot{U}'$ and $\dot{V}'$ in terms of initial $\dot{U}$ and $\dot{V}_1$.

* 6 components

* we have 4 equations and 6 unknowns

* condition 5: collisions are coplanar (of interaction is radial)

  $\therefore \dot{U}'$ is in plane of $\dot{U}$ and $\dot{V}_1$

  and $\dot{U}'$ must be too

* condition 6 depends on nature of collision interaction

  We can use the impact parameter, which will give deflection

  We will assume that a scattering cross-section is given

Consider beam of particles colliding with another beam

\[ \begin{align*}
\dot{U} & \quad n \\
\dot{V} & \quad \text{experiences a flux} \\
\end{align*} \]

\[ I = |\dot{U} - \dot{V}|, \quad \text{of particles from other beam} \]

$\delta n_c$ = $\#$ of collisions$/\dot{U}$/$t$ deflecting from this beam into $d\Omega$

\[ \delta n_c = \sigma(\dot{U}, \dot{V}, \dot{U}', \dot{V}_1) \cdot n \cdot |\dot{U} - \dot{V}|, dn, d\Omega \]

* this is called the differential cross-section
If we assume molecular processes are reversible,
\[ \sigma(\hat{\nu}, \hat{\nu}' \mid \hat{\nu}, \hat{\nu}_1) = \sigma(\hat{\nu}, \hat{\nu}_1 \mid \hat{\nu}', \hat{\nu}') \]

**Collision integral**

First set of particles have \( n \) density \( n = f(x, \nu, t) \, d^3 \nu \)
second particles have \( n_1 = f(x, \nu_1, t) \, d^3 \nu_1 \)

so \( \delta n_c = \sigma(\hat{\nu}, \hat{\nu}_1 \mid \hat{\nu}', \hat{\nu}_1) \, |\hat{\nu} - \hat{\nu}_1| \, f(x, \nu, t) \, f(x, \nu_1, t) \, d^3 \nu \, d^3 \nu_1 \)

\( C_{out} = \text{# of collisions / time in volume } d^3 x \, d^3 \nu, \text{ so} \)
\[ C_{out} = \int d^3 x \, \int d^3 \nu_1 \, \int d\Omega \, \delta n_c \]

\( C_{in} \)
consider reverse collisions between particles w/ velocities \( d^3 \nu \) and those w/ \( d^3 \nu_1 \) where results lie in \( d^3 \nu \) and \( d^3 \nu_1 \)

in this case:
\[ \delta n'_c = \sigma(\hat{\nu}', \hat{\nu}_1' \mid \hat{\nu}, \hat{\nu}_1) \, |\hat{\nu}' - \hat{\nu}_1'| \, f(x, \nu', t) \, f(x, \nu_1', t) \, d\Omega \, d^3 \nu_1' \]

Elastic collisions:
\[ |\hat{\nu} - \hat{\nu}_1| = |\hat{\nu}' - \hat{\nu}_1'| \] (see \( \delta n_v \), e.g.)

**Liouville's theorem**:
\[ d^3 \nu \, d^3 \nu_1 = d^3 \nu' \, d^3 \nu_1' \]
(in 2-particle phase space)

(they are in same \( d^3 x \))
To show relative velocity magnitude is conserved in elastic collisions, we transform to relative and center of mass coordinates

$$\mathbf{\hat{V}} = \mathbf{\hat{v}} - \mathbf{\hat{u}},$$
$$\mathbf{\hat{V}} = \frac{1}{2} (\mathbf{\hat{v}} + \mathbf{\hat{u}}),$$
$$\mathbf{\hat{V}}' = \mathbf{\hat{v}}' - \mathbf{\hat{u}}',$$
$$\mathbf{\hat{V}}' = \frac{1}{2} (\mathbf{\hat{v}}' + \mathbf{\hat{u}}').$$

Then

$$\mathbf{\hat{v}} = \mathbf{\hat{V}} + \frac{1}{2} \mathbf{\hat{V}}'$$
$$\mathbf{\hat{u}} = \mathbf{\hat{V}} - \frac{1}{2} \mathbf{\hat{V}}'$$
$$\mathbf{\hat{v}}' = \mathbf{\hat{V}}' + \frac{1}{2} \mathbf{\hat{V}}'$$
$$\mathbf{\hat{u}}' = \mathbf{\hat{V}}' - \frac{1}{2} \mathbf{\hat{V}}'.$$

Conservation of momentum

$$\mathbf{\hat{v}} + \mathbf{\hat{u}} = \mathbf{\hat{v}}' + \mathbf{\hat{u}}',$$
$$\mathbf{\hat{V}} + \frac{1}{2} \mathbf{\hat{V}}' + \mathbf{\hat{V}} - \frac{1}{2} \mathbf{\hat{V}}' = \mathbf{\hat{V}}' + \frac{1}{2} \mathbf{\hat{V}}' + \mathbf{\hat{V}} - \frac{1}{2} \mathbf{\hat{V}}'$$
$$\therefore \mathbf{\hat{V}} = \mathbf{\hat{V}}'.$$

Conservation of energy

$$|\mathbf{\hat{v}} + \frac{1}{2} \mathbf{\hat{V}}|^2 + |\mathbf{\hat{v}} - \frac{1}{2} \mathbf{\hat{V}}|^2 = |\mathbf{\hat{v}}' + \frac{1}{2} \mathbf{\hat{V}}'|^2 + |\mathbf{\hat{v}} - \frac{1}{2} \mathbf{\hat{V}}'|^2$$

$$|\mathbf{\hat{V}}|^2 + \mathbf{\hat{v}} \cdot \mathbf{\hat{V}} + \frac{1}{4} |\mathbf{\hat{V}}|^2 + |\mathbf{\hat{V}}| - \mathbf{\hat{v}} \cdot \mathbf{\hat{V}} + \frac{1}{4} |\mathbf{\hat{V}}|^2$$
$$= |\mathbf{\hat{V}}'|^2 + \mathbf{\hat{v}}' \cdot \mathbf{\hat{V}} + \frac{1}{4} |\mathbf{\hat{V}}|^2 + |\mathbf{\hat{V}}|^2 - \mathbf{\hat{v}}' \cdot \mathbf{\hat{V}} + \frac{1}{4} |\mathbf{\hat{V}}'|^2$$

the $|\mathbf{\hat{V}}|^2$ and $|\mathbf{\hat{V}}'|^2$ cancel from momentum condition giving

$$|\mathbf{\hat{v}}| = |\mathbf{\hat{v}}'|$$
\[ \sigma \left( \ddot{u}, \dot{u} \right) \left( \ddot{u}, \dot{u} \right) \left| u - \dot{u} \right| \frac{f(x, \dot{u}, t) f(y, \dot{u}, t)}{d \Omega \ d^3 v \ d^3 u} \]

then
\[ C_{\text{in}} = d^3 x \int d^3 v \int d \Omega \ \delta n' \]

together:
\[ \frac{\partial f}{\partial t} + \dot{u} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \nabla \varphi = \int d^3 v \int d \Omega \left| \ddot{u} - \dot{u} \right| \sigma(\Omega) \left( f f' - f f_1 \right) \]

\[ \nabla \cdot \dot{u} = \frac{\vec{F}}{m} \]

non-linear integro-differential equation
Maxwell distribution

— this should be an equilibrium solution of Boltzmann $g$.

Uniform gas ($\frac{df}{dx} = 0$) with negligible external field ($F = 0$)

Note: gas occupying large volume in g-field is stratified and hence not uniform

In equilibrium, $\frac{df}{dt} + \mathbf{u} \cdot \frac{df}{dx} + \frac{F}{m} \frac{df}{du} = 0$

so we just need to look at the collision term, which is zero when

$$f f_1 = f' f'_1$$

note, $f = f(\mathbf{u})$ in equilibrium

Taking log,

$$\log f(\mathbf{u}) + \log f(\mathbf{u}_1) = \log f(\mathbf{u}') + \log f(\mathbf{u}_1')$$

If $X(\mathbf{u})$ is a conserved quantity (in binary collision), then

$$X(\mathbf{u}) + X(\mathbf{u}_1) = X(\mathbf{u}') + X(\mathbf{u}_1')$$

comparing, we need

$$\log f(\mathbf{u}) = C_0 + \sum \mathbf{c}_r X_r(\mathbf{u})$$

$X_r$ are all independently conserved quantities
Consider conservation of (3) momenta and energy.

\[ \log f(\hat{u}) = C_0 + C_1 u^2 + C_2 \cdot \hat{u}_0 \]

which we write as

\[ \log f(\hat{u}) = -B(\hat{u} - \hat{u}_0)^2 + \log A \]

\[ = -B|\hat{u}|^2 + B|\hat{u}_0|^2 + 2B\hat{u}_0 \cdot \hat{u} + \log A \]

\[ C_0 = \log A - B|\hat{u}_0|^2 \]

\[ C_1 = -B \]

\[ C_2 = 2B\hat{u}_0 \]

\[ \]

then we have

\[ f(\hat{u}) = A e^{-B(\hat{u} - \hat{u}_0)^2} \]

\[ \]

\[ n = \int d^3 u f(\hat{u}) = A \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz e^{-B[(u_x - u_{x0})^2 + (u_y - u_{y0})^2 + (u_z - u_{z0})^2]} \]

since

\[ \int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}} \]

we pick up 3 of these,

\[ n = A \left( \frac{\pi}{B} \right)^{3/2} \rightarrow A = \left( \frac{B}{\pi} \right)^{3/2} n \]
This almost looks Maxwellian, but we have $\tilde{U}_0$.

Average velocity:

$$<\tilde{u}> = \frac{\int_{-\infty}^{\infty} d^3\tilde{u} \tilde{u} f(\tilde{u})}{\int_{-\infty}^{\infty} d^3\tilde{u} f(\tilde{u})}$$

Normalization

$$= \frac{A}{n} \int_{-\infty}^{\infty} d^3\tilde{u} \tilde{u} e^{-\frac{B(\tilde{u}-\tilde{u}_0)^2}{2}} = \frac{A}{n} \int d^3\tilde{u} (\tilde{u} + \tilde{u}_0) e^{-\frac{B\tilde{u}^2}{2}}$$

$$= \frac{A}{n} \int_{-\infty}^{\infty} d^3\tilde{u} \tilde{u} e^{-B\tilde{u}^2} + \frac{A}{n} \int \tilde{u}_0 e^{-B\tilde{u}_0^2} d^3\tilde{u}$$

$$= \frac{A}{n} \tilde{u}_0 \left( \frac{\pi}{B} \right)^{3/2} = \tilde{u}_0$$

So $\tilde{u}_0$ is just average velocity.
What is \( \int_{-\infty}^{\infty} x e^{-ax^2} \, dx \)?

\[
\int_{-\infty}^{0} x e^{-ax^2} \, dx + \int_{0}^{\infty} x e^{-ax^2} \, dx
\]

\[x \cdot \frac{d}{dx} = -x\]

\[d\left(\frac{x}{2}\right) = -dx\]

\[
\int_{-\infty}^{0} \left(-\frac{x}{2}\right) e^{-a\frac{x^2}{2}} \, dx + \int_{0}^{\infty} x e^{-ax^2} \, dx
\]

\[= -\int_{0}^{\infty} \frac{x}{2} e^{-a\frac{x^2}{2}} \, dx + \int_{0}^{\infty} x e^{-ax^2} \, dx = 0\]
So in frame moving at \( \tilde{v}_0 \), we are Maxwellian.

We take \( B = \frac{m}{2k_B T} \)

then

\[ f(\tilde{v}) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left[ -\frac{m (\tilde{v} - \tilde{v}_0)^2}{2k_B T} \right] \]

so the Maxwellian is the equilibrium solution.
For a conserved quantity,
\[ x + x_1 = x' + x_1' \]

We consider what the Boltzmann equation says for \( x \).

Multiply the Boltzmann equation by \( x \) and integrate over \( d^3 \mathbf{u} \)

\[
\int d^3 \mathbf{u} \, x \left( \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f + \frac{F}{m} \cdot \nabla_x f \right)
\]

\[
= \int d^3 \mathbf{u} \int d^3 \mathbf{u}' \int d \mathbf{\sigma} \, |\mathbf{u} - \mathbf{u}'| \sigma(\mathbf{\sigma}) \left( f f' - f f_1' \right) \]

Look at the right-hand side (the collision integral).

- \( u \) and \( u_1 \) enter symmetrically everywhere except \( x \)
- they are dummy variables here (since we are integrating them out)

This means we could substitute \( x \rightarrow \frac{1}{2} (x + x_1) \) and we'd get the same result.

The fact that \( |\mathbf{u}' - \mathbf{u}_1'| = |\mathbf{u} - \mathbf{u}_1| \), \( \sigma \) is reversible, and

\( d^3 \mathbf{u} \, d^3 \mathbf{u}' = d^3 \mathbf{u}' \, d^3 \mathbf{u} \),

mean that we can rename the primed variables to be unprimed and vice versa.

We'll pick up a sign change from \( f f' - f f_1' \).

Together, this suggests we can replace

\[ x \rightarrow \frac{1}{4} (x + x_1 - x' - x_1') \]

and we'd get the same answer.

But this is \( 0! \) because of conservation of \( x \).
This basically says that collisions cannot contribute to $\frac{Dx}{Dt}$ if $x$ is a conserved quantity.

(See Sho at bottom of p. 19)
For conserved quantity
\[ \chi + \chi_1 = \chi' + \chi'_1, \]
multiply Boltzmann by \( \chi \) and integrate over \( d^3 x \)
\[
\int d^3 x \chi \left( \frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{E}{m} \cdot \nabla j_f \right)
= \int d^3 x_1 \int d^3 x \int d\Omega \left| \tilde{v} - \tilde{v}_1 \right| \sigma(\Omega) (f'f'_1 - ff_1) \chi
\]
now consider replacing \( \tilde{v} \) and \( \tilde{v}_1 \), (we are integrating over both)
\[
\text{RHS} = \frac{1}{2} \int d^3 v \int d^3 v_1 \int d\Omega \left| \tilde{v} - \tilde{v}_1 \right| \sigma(\Omega) (f'f'_1 - ff_1) \chi
+ \frac{1}{2} \int d^3 v \int d^3 v_1 \int d\Omega \left| \tilde{v}_1 - \tilde{v} \right| \sigma(\Omega) (f'f'_1 - f'_1 f) \chi_1
= \frac{1}{2} \int d^3 v \int d^3 v_1 \int d\Omega \left| \tilde{v} - \tilde{v}_1 \right| \sigma(\Omega) (f'f'_1 - ff_1) \left[ \chi + \chi_1 \right]
\]
since we are reversible, we must be symmetric w/ \((v,v_1) \rightarrow (v_1,v)\)
\[
\text{RHS}' = \frac{1}{2} \int d^3 v' \int d^3 v_1' \int d\Omega \left| \tilde{v}' - \tilde{v}_1' \right| \sigma(\Omega)(ff_1 - f'f_1') \left[ \chi' + \chi'_1 \right]
= \left| \tilde{v} - \tilde{v}_1 \right| (f'f'_1 - ff_1)
\]
by Liouville's theorem
\[
\text{RHS} = \frac{1}{2} \int d^3 v \int d^3 v_1 \int d\Omega \left| \tilde{v} - \tilde{v}_1 \right| \sigma(\Omega) (f'f'_1 - ff_1) \left( \chi + \chi_1, \chi' + \chi'_1 \right)
= \text{0 by conservation} \]
That leaves us with
\[
\int d^3u \chi \left( \frac{\partial f}{\partial t} + u \cdot \nabla f + \frac{F}{m} \cdot \nabla u f \right) = 0
\]
writing out:
\[
\int d^3u \chi \frac{\partial f}{\partial t} + \int d^3u \chi u \cdot \nabla f + \int d^3u \chi \frac{F}{m} \cdot \nabla u f = 0
\]
we'll integrate by parts
\[
\int d^3u \left[ \frac{\partial f x}{\partial t} - f \frac{\partial x}{\partial t} \right] \rightarrow 0
\]
\[
\text{later we'll take } x = x(u) \text{ (see, e.g. Shu)} \quad \text{Eq 2.18}
\]
\[
+ \int d^3u \left[ \frac{\partial}{\partial x_i} \left( x U_i f \right) - \delta_i^j f \frac{\partial x_i}{\partial x_j} \left( u x \right) \right]
\]
\[
\text{we can pull } u \text{ out}
\]
\[
+ \int d^3u \left[ \frac{\partial}{\partial u_i} \left( \frac{F_i}{m} x f \right) - \frac{1}{m} f x \frac{\partial F_i}{\partial u_i} - \frac{F_i}{m} f \frac{\partial x}{\partial u_i} \right]
\]
\[
\text{note } A \rightarrow this \text{ term is zero so long as } f \rightarrow 0 \text{ at } \infty
\]
Now we have
\[
\frac{\partial}{\partial t} \int d^3u f x + \frac{\partial}{\partial x_i} \int d^3u x U_i f - \int d^3u f u \frac{\partial x}{\partial x_i}
\]
\[
- \frac{1}{m} \int d^3u \frac{\partial x}{\partial u_i} F_i f - \frac{1}{m} \int d^3u x \frac{\partial F_i}{\partial u_i} f = 0
\]
Now define average as
\[
\langle Q \rangle = \frac{1}{n} \int d^3v \ Q f
\]
:: \[ \int d^3v \ Q f = n \langle Q \rangle \]
and we have
\[
\frac{\partial}{\partial t} \left( n \langle x \rangle \right) + \frac{\partial}{\partial x_i} \left( n \langle u_i x \rangle \right) - n \langle u \frac{\partial x}{\partial x_i} \rangle - \frac{n}{m} \langle \dot{x} \frac{\partial x}{\partial u_i} \rangle - \frac{n}{m} \langle x \frac{\partial F_i}{\partial u_i} \rangle = 0
\]
we'll often assume that $F$ is independent of velocity. Then the last term vanishes, and $F$ can be pulled out of the average (since $u$ is an integral over $u$)
\[
\frac{\partial}{\partial t} \left( n \langle x \rangle \right) + \frac{\partial}{\partial x_i} \left( n \langle u_i x \rangle \right) - n \langle u \frac{\partial x}{\partial x_i} \rangle - n \frac{F_i}{m} \langle \frac{\partial x}{\partial u_i} \rangle = 0
\]