Accretion disks

Re \gg 1 in most astrophysical systems (U, L large) this means that viscosity only important in boundary layers

most astrophysical systems do not have boundaries, so viscosity is usually ignored.

However, for accretion disks, this is not the case.

Accretion disks convert gravitational potential energy into heat + radiation.

Drop mass from height h in g field g.

\( \Delta \): gravitational PE is mgh, converted into KE then (on hitting the ground) heat & sound.

For mass m dropped onto star of mass M, radius a,

\[
\Delta = \frac{GMm}{a} = \frac{GM}{ac^2} mc^2 = \epsilon mc^2
\]

\( \epsilon \) is the fraction of rest mass.

\( \epsilon \) can be 0.15 for a neutron star.

Will discuss spherical accretion.

But for accretion, conservation of angular momentum leads to a disk.
Shakura & Sunyaev (1973) gave formalism for thin accretion disks.

Consider matter orbiting mass $M$ in nearly circular orbit

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

take

$$v = \Omega r$$

$$\Omega^2 r = \frac{GM}{r^2} \rightarrow \Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

(This is like Kepler’s law)

We call this Keplerian.

If a disk has this structure, then there is a lot of shear — we expect viscosity to be important

- Angular momentum transferred from faster-moving inner regions of disk to slower-moving outer regions
- Inner material loses angular momentum — spirals inward

Viscosity determines the rate of radial inflow of matter and the rate of conversion of GPE.
Disk dynamics: \( \ddot{\mathbf{v}} = v_r \hat{\mathbf{r}} + v_\theta \hat{\theta} \)

\( |v_r| \ll |v_\theta| \)

Azimuthal symmetry: \( \partial / \partial \theta = 0 \)

Our momentum equation is:

\( \rho \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = \mu \nabla^2 \mathbf{v} \)

We want the \( \hat{\theta} \) component.

Note: no pressure gradient in \( \theta \), so \( \nabla p \cdot \hat{\theta} = 0 \)

For \( (\nabla \cdot \mathbf{v}) \mathbf{v} \), we need to be careful about differentiating the unit vectors, so we write

\( (\nabla \cdot \mathbf{v}) \mathbf{v} = \frac{1}{2} \nabla (\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v}) \)

We want the \( \hat{\theta} \) component, but the first term is \( \partial / \partial \theta = 0 \), so only the second term contributes.
What is $-\mathbf{v} \times (\nabla \times \mathbf{v})$?

$$\mathbf{v} = v_r \hat{r} + v_\theta \hat{\theta}$$

$$\nabla \times \mathbf{v} = -\frac{\partial v_\theta}{\partial z} \hat{r} + \frac{\partial v_r}{\partial z} \hat{\theta} + \left[ \frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \hat{z}$$

then

$$-\mathbf{v} \times (\nabla \times \mathbf{v}) = -\begin{vmatrix}
\hat{r} & \hat{\theta} & \hat{z} \\
v_r & v_\theta & 0 \\
-\frac{\partial v_\theta}{\partial z} & \frac{\partial v_r}{\partial z} & \frac{1}{r} \frac{\partial (r v_\theta)}{\partial r}
\end{vmatrix}$$

$$= -v_\theta \frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \hat{r} + \frac{v_r}{r} \frac{\partial (r v_\theta)}{\partial r} \hat{\theta}$$

$$- (v_r \frac{\partial v_r}{\partial z} + v_\theta \frac{\partial v_\theta}{\partial z}) \hat{z}$$

We only care about $\hat{\theta}$, so

$$-\mathbf{v} \times (\nabla \times \mathbf{v}) \cdot \hat{\theta} = \frac{v_r}{r} \frac{\partial (r v_\theta)}{\partial r} = v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r}$$

Our advective term for $\hat{\theta}$ is

$$\rho \left[ \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} \right]$$
Now $\nabla^2 v$ for the viscous term.

Again, we are careful about differentiating unit vectors, so we write

$$\nabla^2 v = \nabla (\nabla \cdot v) - \nabla \times (\nabla \times v)$$

For $\hat{\theta}$, this term vanishes, since it is $\frac{\partial}{\partial \theta}$

that leaves $\nabla^2 v = -\nabla \times (\nabla \times v)$

We already computed $\nabla \times v = -\frac{\partial v_\theta}{\partial z} \hat{r} + \frac{\partial v_r}{\partial z} \hat{\theta} + \frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \hat{\theta}$

now $-\nabla \times (\nabla \times v) \cdot \hat{\theta} = - \left[ -\frac{\partial^2 v_\theta}{\partial z^2} - \frac{2}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) \right]$

$$= \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial^2 (r v_\theta)}{\partial r^2} - \frac{1}{r^2} \frac{\partial (r v_\theta)}{\partial r}$$

$$= \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r \frac{\partial v_\theta}{\partial r} + v_\theta \right] - \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}$$

$$= \frac{\partial^2 v_\theta}{\partial z^2} + \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}$$
This gives

\[ \rho \left[ \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} \right] = \mu \left[ \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{1}{v} \frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta}{r^2} \right] \]

Continuity is straightforward:

\[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial }{\partial r} (r \rho v_r) = 0 \]
Note: we assumed $\mu = \text{constant}$ since $J = \frac{\mu}{\rho}$ this means we assumed $\mu = \rho \Omega = \text{const}$

We'll try to figure out the viscous stress another way

Define $\int \rho \, dz = \Sigma$ (surface density)

and take $v_r$ and $v_\theta$ to be independent of $z$ (but $\rho = \rho(z)$)

now integrate both sides over $dz$

$$\int dz \left[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) \right] = 0$$

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) = 0 \quad (i)$$

$z$ integral commutes w/ $\frac{\partial}{\partial r}$

momentum:

$$\int dz \rho \left[ \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} \right] = \int dz \mu \cdots$$

$$\Sigma \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} \right) = \text{viscous terms} \quad (ii)$$

do $rv_\theta \cdot (i) + r \cdot (ii)$

$$rv_\theta \frac{\partial \Sigma}{\partial t} + v_\theta \frac{\partial}{\partial v} (r \Sigma v_r) + r \Sigma \frac{\partial v_\theta}{\partial t} + r \Sigma v_r \frac{\partial v_\theta}{\partial r} + \kappa \Sigma \frac{v_r v_\theta}{\kappa}$$

defining $v_\theta = \frac{s}{2} r$

$$= \frac{\eta}{\kappa} \text{viscous stuff}$$
time derivatives:

\[ \partial_t \left( \rho \Sigma \right) = \partial_t \left( \rho v_\theta \Sigma \right) = \rho v_\theta \frac{\partial \Sigma}{\partial t} + \Sigma \frac{\partial}{\partial t} \left( \rho v_\theta \right) \]

\( r \) is a geometric term

\[ = \rho v_\theta \frac{\partial \Sigma}{\partial t} + \Sigma \frac{\partial}{\partial t} \left( \rho v_\theta \right) \]

these are the time derivatives

Now consider

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( \Sigma r^2 \rho v_r \right) \]

\[ = \frac{\rho v_\theta}{r} \frac{\partial}{\partial r} \left( \Sigma r v_r \right) + \frac{\Sigma v_r}{r} \frac{\partial}{\partial r} \left( \rho v_\theta r \right) \]

\[ = \rho v_\theta \frac{\partial}{\partial r} \left( r \Sigma v_r \right) + \Sigma v_r \left( v_\theta + r \frac{\partial v_\theta}{\partial r} \right) \]

these are the space derivatives

We have

\[ \frac{\partial}{\partial t} \left( \Sigma r^2 \Omega \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \Sigma r^2 v_r \right) = \dot{\Omega} \]

w/ \( \Omega = \frac{v_\theta}{r} \)

angular velocity

Note: \( \Sigma r^2 \Omega \cdot 2\pi r \, dr \) is angular momentum associated w/ annular ring between \( r \) and \( r + dr \)

\[ ( L = m v_\theta r = m \Sigma 2r^2 \Omega \) but \( m = 2\pi r \, dr \, \Sigma \)

\[ \therefore L = \Sigma r^2 \Omega \cdot 2\pi r \, dr \]

this is an evolution equation for angular momentum
This equation is then

\[
\frac{\partial \text{LA}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{L_r}{A} \right) = G
\]

\[
\text{divergence of angular momentum flux}
\]

\[
\text{multiplying by } 2\pi r dr, \text{ then this gives evolution of angular momentum in annulus. Looking at the RHS,}
\]

\[
G \cdot 2\pi r dr = G(r + dr) - G(r)
\]

\[
\uparrow \text{viscous torque}
\]

\[
G = \frac{1}{2\pi r} \frac{dG}{dr}
\]

and now we need to find the viscous torque

Consider \[
\frac{d\Omega}{dr} = \frac{d}{dr} (r \Omega) = \Omega + r \frac{d\Omega}{dr}
\]

\[
\uparrow \text{pure rotation}
\]

\[
\uparrow \text{shear}
\]

viscous stress is \[
\mu r \frac{d\Omega}{dr}
\] (force/unit area)

viscous torque/unit area: \[
r \cdot \text{viscous stress}
\]
then viscous torque is

\[
G(r) = \int r d\theta \int dz \mu r^2 \frac{d\Omega}{dr} = 2\pi \int \frac{d\Omega}{dr}
\]

\[
\uparrow \frac{\mu}{\rho} = p d
\]
Then 
\[
\mathcal{M} = \frac{1}{2\pi r} \frac{dE}{dr} = \frac{1}{2\pi r} \cdot 2\pi r \cdot \int \sum r^3 v \frac{d\Omega}{dr}
\]

\[
= \frac{1}{2\pi r} \cdot 2\pi \frac{d}{dr} \left( v r^3 \sum \frac{d\Omega}{dr} \right)
\]

and we have

\[
\frac{2}{\dot{\theta}} \left( r^2 \dot{\Omega} \right) + \frac{1}{r} \frac{d}{dr} \left( r^3 \dot{v} \right) = \frac{1}{r} \frac{d}{dr} \left( v \sum r^3 \frac{d\Omega}{dr} \right)
\]

together w/ 

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{d}{dr} \left( r \Sigma v_r \right) = 0
\]

we have a complete system
Let's rewrite

note: \( \frac{\partial r}{\partial t} = 0 \)

Also if we take \( \Omega \) to be Keplerian, then \( \frac{\partial \Sigma}{\partial t} = 0 \) (since \( \Sigma \) depends only on \( r \))

Then our momentum equation is

\[
\frac{d}{dt} \left( \sum r^2 \Omega r v_r \right) = \int \frac{d}{dr} \left( \sum r^2 \frac{d \Omega}{dr} \right)
\]

Insert continuity, \( -\frac{1}{r} \frac{\partial}{\partial r} \left( r \sum v_r \right) = \frac{\partial \Sigma}{\partial t} \)

\[
= r \Omega \frac{\partial}{\partial r} \left( r \sum v_r \right) + \int \frac{d}{dr} \left( \sum r^2 \Omega v_r \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( \sum r^2 \frac{d \Omega}{dr} \right)
\]

Expand this

\[
- r \Sigma \frac{\partial}{\partial r} \left( r \sum v_r \right) + r \Omega \frac{\partial}{\partial r} \left( \sum r v_r \right) + \Sigma v_r \frac{\partial}{\partial r} (\Omega r^2)
\]

\[
= \frac{1}{r} \frac{\partial}{\partial r} \left( \sum r^2 \frac{d \Omega}{dr} \right)
\]

\[
\therefore \ v_r = \frac{1}{r \Sigma} \frac{1}{\partial r(\Omega r^2)} \frac{\partial}{\partial r} \left( \sum r^2 \frac{d \Omega}{dr} \right)
\]
Now let's take $\Omega = GM r^{-3/2}$

some derivatives:

$$\frac{2}{\partial r} (\Omega r^2) = \frac{1}{2} GM r^{-1/2}$$

$$\frac{2}{\partial r} \Omega = -\frac{3}{2} GM r^{-5/2}$$

so

$$v_r = \frac{1}{r \Sigma} \frac{\Sigma r^{1/2}}{GM} \frac{2}{\partial r} \left( \nu \Sigma r^{3} \left( -\frac{3}{2} GM r^{-5/2} \right) \right)$$

$$= -\frac{3}{\sqrt{r} \Sigma} \frac{2}{\partial r} \left( \nu \Sigma r^{1/2} \right)$$

that's the radial velocity in the disk— notice its dependence on viscosity, $\nu$. 

Now back to continuity:

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \Sigma v_r \right) = 0
\]

Inserting our \( v_r \), we have the thin disk evolution equation (diffusion)

\[
\frac{\partial \Sigma}{\partial t} + \frac{3}{r} \frac{\partial}{\partial r} \left[ r^{1/2} \frac{\partial}{\partial r} \left( \sqrt{\Sigma} r^{1/2} \right) \right] = 0 \quad \text{(Choudhuri eq 5.49)}
\]

For constant \( \Sigma \), this can be solved via separation of variables.
Solution for a ring at radius \( r = r_0 \):

\[
\Sigma(r, t=0) = \frac{m}{2\pi r_0} \delta(r - r_0)
\]

(total mass in ring is \( m \))

Solution is \( \Sigma(x, t) = \frac{m}{\pi r_0^2} \frac{1}{x} \frac{1}{\tau} e^{-(1+x^2)/\tau} \) \( I_0 \left( \frac{2x}{\tau} \right) \)

\( x = \frac{r}{r_0}, \tau = \frac{12 \pi t}{r_0^2} \)

\( I_0 \) modified Bessel function
Steady disk

What are \( r \) and \( z \) components of Navier–Stokes?

\( z \) is easy, since there is no velocity

\[
q \cdot \hat{z} = - \frac{GM}{r^2} \hat{\theta} \sin \theta
\]

\[
= - \frac{GM}{r^2} \frac{z}{r}
\]

(assuming \( z \ll r \))

Now what about the radial?

When we computed \( (v \cdot \nabla) v = \frac{1}{2} \nabla (v \cdot v) - v \times (\nabla \times v) \)

we had the \( \hat{r} \) term for the cross product.

Now the first term is not zero

\[
\frac{1}{2} \nabla \left( v_r^2 + v_\theta^2 \right) \cdot \hat{r} = v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_\theta}{\partial r}
\]

\[- \cdot v \times (\nabla \times v) \cdot \hat{r} = - v_\theta \frac{1}{r} \frac{\partial (rv_\theta)}{\partial r} = - v_\theta \frac{\partial v_\theta}{\partial r} - \frac{v_\theta^2}{r}
\]

\[
\therefore \text{ we have}
\]

\[
v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} = - \frac{1}{p} \frac{\partial p}{\partial r} - \frac{GM}{r^2}
\]

and \( z \):

\[
- \frac{1}{p} \frac{\partial p}{\partial z} - \frac{GM z}{r^3} = 0
\]
for disk of thickness $h$, $\frac{dp}{dz} \sim \frac{P}{h}$

$\therefore \frac{P}{\rho h} \sim \frac{GMh}{r^3}$

or $(\frac{h}{r})^2 \sim \frac{rp}{GM\rho}$

$h \ll r$ implies $\frac{rp}{GM\rho}$ also shows (for ideal gas $\frac{p}{\rho}$ = const) $h \sim r^3$

In the radial equation,

$\frac{1}{\rho} \frac{dp}{dv} - \frac{GM}{r^2} \sim \frac{1}{\rho} \frac{p}{h} - \frac{GM}{r^2}$

ratio: $\frac{|\frac{1}{\rho} \frac{p}{r}|}{|GM/r^2|} \sim \frac{rp}{GM\rho} \sim (\frac{h}{r})^2 \ll 1$

so the pressure gradient is insignificant

Also we expect $|v_r| \ll (v_\theta)$, so we can approximate the radial equation as

$\frac{v_\theta^2}{r} = \frac{GM}{r^2}$ (this is keplerian)

this demonstrates that keplerian motion only holds for thin disks
So for a thin disk, we don't need the radial N-S equation (it is just Keplerian).

Then we go back to
\[
\frac{\partial}{\partial t} \left( \rho \Sigma \right) + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) = 0
\]

getting rid of time derivatives (steady), we can integrate once easily:

\[
\begin{align*}
  r \Sigma v_r &= \alpha \\
  \Sigma r^3 \omega v_r - \int \Sigma r^3 \frac{d\omega}{dr} &= \beta
\end{align*}
\]

(\alpha, \beta \text{ are integration constants})

Now in steady state, we have a mass flow
\[
\dot{m} = -2\pi r \Sigma v_r
\]

\[
\therefore \alpha = - \frac{\dot{m}}{2\pi}
\]

For the other constant, we note that at the surface of the star, \( r = r_* \), \( \frac{d\omega}{dr} = 0 \) (we need to move with the star), so

\[
\begin{align*}
  \beta &= \Sigma r_*^3 \omega v_r \\
  \text{but } v_r &= - \frac{\dot{m}}{2\pi r \Sigma} \\
  \therefore \beta &= - \frac{\dot{m}}{2\pi} r_*^2 \omega
\end{align*}
\]
Using $\Omega = (GM)^{1/2} r^{-3/2}$, $\beta = -(GM)^{1/2} \frac{m}{2\pi} r_*^{1/2}$

then our original steady state momentum is

$$\sum r^3 \Omega v_r - \nu \sum r^3 \frac{d\Omega}{dr} = - (GM)^{1/2} \frac{m}{2\pi} r_*^{1/2}$$

Substituting in for $v_r$ and $\Omega$ ($w/ \frac{d\Omega}{dr} = \frac{3}{2} (GM)^{1/2} r^{-5/2}$)

$$\sum r^3 r^{-3/2} \left( - \frac{m}{2\pi r_* \Delta} \right) - \nu \sum r^3 \left( - \frac{3}{2} r^{-5/2} \right) = - \frac{m}{2\pi} r_*^{1/2}$$

$$r_*^{1/2} \dot{m} + \nu \sum 3\pi r_*^{1/2} = \dot{m} r_*^{1/2}$$

$$\therefore \nu \Sigma = \frac{m}{3\pi} \left( 1 - \left( \frac{r_*}{r} \right)^{1/2} \right)$$

This shows that the mass flow is proportional to $\nu$. 