1. In our discussion on shocks, we derived an expression for the shock speed for a right-moving shock as

\[ S = u_R + c_R \sqrt{\left( \frac{\gamma + 1}{2\gamma} \right) \left( \frac{p_*}{p_R} \right) + \left( \frac{\gamma - 1}{2\gamma} \right)} \]  

(1)

by moving into the frame of the shock

\[ \hat{u}_* = u_* - S \]  

(2)

\[ \hat{u}_R = u_R - S \]  

(3)

and applying the jump conditions:

\[ \rho_* \hat{u}_* = \rho_R \hat{u}_R \]  

(4)

\[ \rho_* \hat{u}_*^2 + p_* = \rho_R \hat{u}_R^2 + p_R \]  

(5)

\[ \hat{u}_* \left( \rho_* e_* + \frac{1}{2} \rho_* \hat{u}_*^2 + p_* \right) = \hat{u}_R \left( \rho_R e_R + \frac{1}{2} \rho_R \hat{u}_R^2 + p_R \right) \]  

(6)

Here, we will derive some related expressions that will be very useful later.

- We begin by seeking an expression for the density ratio in terms of the pressure ratio. Using Eqs. 4 and 5, eliminate the velocities in the energy expression in favor of the pressures and densities to get:

\[ e_* - e_R = \frac{1}{2} \left( p_* + p_R \right) \frac{\rho_* - \rho_R}{\rho_* \rho_R} \]  

(7)

This relation is called the Hugoniot equation for the shock. It relates all the points in the \((\rho, p)\) plane that can be connected by a shock wave.

- Now, assuming a gamma-law equation of state, solve for the density ratio in terms of the pressure ratio to yield:

\[ \frac{\rho_*}{\rho_R} = \frac{\frac{p_*}{p_R} \frac{\gamma - 1}{\gamma + 1} + 1}{\frac{p_*}{p_R} \frac{\gamma - 1}{\gamma + 1}} \]  

(8)

- Now we want to compute the post-shock velocity, \(u_*\). From the Eq. 4 and Eqs. 2 and 3, show that

\[ u_* = \left( 1 - \frac{\rho_R}{\rho_*} \right) S + \frac{\rho_R}{\rho_*} u_R \]  

(9)

and substitute in Eq. 1 and Eq. 8 to get the expression for the post-shock speed in terms of the pressure ratio:

\[ u_* = u_R - \frac{2c_R}{\sqrt{2\gamma(\gamma - 1)}} \frac{1 - \frac{p_*}{p_R}}{\sqrt{\frac{\gamma - 1}{\gamma + 1} \frac{p_*}{p_R}}} \]  

(10)

We will make use of this expression in class when computing the solution to the Euler equations.
2. (from Choudhuri) The jet coming out of a galaxy pushes the surrounding gas away and creates a channel through which the light gas inside the jet flows. The pressure inside the jet has to be equal to the pressure outside. Assuming the flow inside the jet to be one-dimensional and adiabatic, show that the local Mach number at a place of pressure \( p \) is given by

\[
M^2 = \frac{2}{\gamma - 1} \left[ \left( \frac{p_0}{p} \right)^{(\gamma - 1)/\gamma} - 1 \right]
\]  

(11)

where \( p_0 \) is the stagnation pressure, i.e. the pressure that gas would have at a point if the velocity were zero there. Show that the cross-section \( A \) of the jet channel varies with the pressure \( p \) as

\[
A \propto \left( \frac{p_0}{p} \right)^{(1/2)(1+1/\gamma)} \left[ \left( \frac{p_0}{p} \right)^{(\gamma - 1)/\gamma} - 1 \right]^{-1/2}
\]  

(12)
1. We want to find some shock jump relations

a. We want \( \frac{p_\star}{p_R} = f \left( \frac{p_\star}{p_R} \right) \)

starting w/ \( \hat{u}_\star \left( \frac{p_\star e_\star + \frac{1}{2} \frac{p_\star}{p_R} \frac{\hat{u}_\star}{p_R}^2}{p_\star} = \hat{u}_R \left( \frac{p_R e_R + \frac{1}{2} \frac{p_R}{p_R} \frac{\hat{u}_R}{p_R}^2}{p_R} \right) \)

factor out density and use \( p_\star \hat{u}_\star = p_R \hat{u}_R \)

\( e_\star + \frac{1}{2} \frac{\hat{u}_\star}{p_R}^2 + \frac{p_\star}{p_R} = e_R + \frac{1}{2} \frac{\hat{u}_R}{p_R}^2 + \frac{p_R}{p_R} \)

w/ \( h = e + \frac{p_\star}{p_R} \)

\[ h_\star + \frac{1}{2} \frac{\hat{u}_\star}{p_R}^2 = h_R + \frac{1}{2} \frac{\hat{u}_R}{p_R}^2 \]  \( \text{eq.} \)

Now we write \( h_\star - h_R = \frac{\hat{u}_\star}{p_R} \left( \frac{p_\star}{p_R} \right) \)

beginning w/ momentum:

\[ p_\star \frac{\hat{u}_\star}{p_R}^2 = p_R \frac{\hat{u}_R}{p_R}^2 + p_R - p_\star = \left( p_\star \frac{\hat{u}_\star}{p_R} \right) \left( \frac{p_R \frac{\hat{u}_R}{p_R}^2}{p_R} \right) + p_R - p_\star \]

so \[ \hat{u}_\star \left( \frac{p_\star}{p_R} \right) = p_R - p_\star \]

\[ \frac{\hat{u}_\star}{p_R} = \left( \frac{p_R}{p_\star} \right) \frac{p_R - p_\star}{p_R - p_\star} \]

similarly, \( p_R \frac{\hat{u}_R}{p_R}^2 = (p_R \frac{\hat{u}_\star}{p_R}) \frac{\hat{u}_\star}{p_R} + p_\star - p_\star = (p_R \frac{\hat{u}_R}{p_R}) \frac{p_R \frac{\hat{u}_R}{p_R}^2}{p_\star} + p_\star - p_R \)

\[ \hat{u}_R \left( \frac{p_R}{p_\star} \right) = p_\star - p_R \]

\[ \frac{\hat{u}_R}{p_R} = \left( \frac{p_R}{p_\star} \right) \frac{p_\star - p_R}{p_\star - p_R} \]
Putting these into our enthalpy equation:

\[ h_\bullet - h_R = \frac{1}{2} (\hat{\mathbf{v}}_R^2 - \hat{\mathbf{v}}_k^2) \]

\[ = \frac{1}{2} \left( \frac{P_R - P_k}{P_R - P_k} \right) \left( \frac{P_k}{P_R} - \frac{P_k}{P_k} \right) \]

\[ = \frac{1}{2} (P_k - P_R) \left( \frac{P_k}{P_R} \right)^{-1} \left( \frac{P_k^2 - P_R^2}{P_k P_R} \right) \]

\[ = \frac{1}{2} (P_k - P_R) \frac{P_k + P_R}{P_k P_R} \quad \text{this is what we wanted in b} \]

Then in terms of \( e \):

\[ e_\bullet - e_R = \frac{1}{2} (P_k - P_R) \frac{P_k + P_R}{P_k P_R} + \frac{P_k}{P_R} - \frac{P_k}{P_k} \]

\[ = \frac{1}{P_k P_R} \left[ \frac{1}{2} P_k P_k + \frac{1}{2} P_k P_R - \frac{1}{2} P_R P_k - \frac{1}{2} P_k P_R + P_k P_k - P_k P_R \right] \]

\[ = \frac{1}{P_k P_R} \left[ \frac{1}{2} (P_k P_k - P_k P_R + P_R P_k - P_k P_R) \right] \]

\[ = \frac{1}{2} (P_k + P_R) \frac{(P_k - P_R)}{P_k P_R} \]
Now, using the EOS

\[ e_* - e_R = \frac{1}{\gamma - 1} \left( \frac{p_*}{p_*^\gamma} - \frac{p_R}{p_R^\gamma} \right) = \frac{1}{2} \left( \frac{p_*}{p_*^\gamma} + \frac{p_R}{p_R^\gamma} \right) \left( \frac{p_*}{p_*^\gamma} - \frac{p_R}{p_R^\gamma} \right) \]

\[ \frac{1}{\gamma - 1} \frac{p_*}{p_*^\gamma} \left( 1 - \frac{p_R}{p_*^\gamma} \frac{p_*^\gamma}{p_R^\gamma} \right) = \frac{1}{\gamma - 1} \frac{p_*}{p_*^\gamma} \left( 1 + \frac{p_R}{p_*^\gamma} \right) \left( \frac{p_*^\gamma}{p_R^\gamma} - 1 \right) \]

defining \( \alpha = \frac{p_*}{p_R} \) \( \beta = \frac{p_*^\gamma}{p_R^\gamma} \)

\[ \frac{1}{\gamma - 1} \left( 1 - \frac{\beta}{\alpha} \right) = \frac{1}{\gamma - 1} \left( 1 + \frac{\beta}{\alpha} \right) \left( \beta - 1 \right) = \frac{1}{\gamma - 1} \left( \beta - 1 + \frac{\beta}{\alpha} - \frac{1}{\alpha} \right) \]

\[ 1 - \frac{\beta}{\alpha} = \frac{\gamma - 1}{2} \beta - \frac{\gamma - 1}{2} - \frac{\gamma - 1}{2} \beta - \frac{\gamma - 1}{2} \alpha \]

get all \( \beta \) on the left

\[ -\frac{\beta}{\alpha} - \frac{\gamma - 1}{2} \beta - \frac{\gamma - 1}{2} \alpha = -1 - \frac{\gamma - 1}{2} - \frac{\gamma - 1}{2} \alpha \]

\[ \beta \left( -\frac{1}{\alpha} - \frac{\gamma - 1}{2} - \frac{\gamma - 1}{2} \alpha \right) = -1 - \frac{\gamma - 1}{2} - \frac{\gamma - 1}{2} \alpha \]

\[ \beta \left( \frac{-2 - (\gamma - 1) \alpha - (\gamma - 1)}{2 \alpha} \right) = \frac{-2 \alpha - (\gamma - 1) \alpha - (\gamma - 1)}{2 \alpha} \]

\[ \beta \left( -2 - \gamma \alpha + \alpha - \gamma + 1 \right) = -2 \alpha - \gamma \alpha + \alpha - \gamma + 1 \]

\[ \beta \left( - (\gamma + 1) - \alpha (\gamma - 1) \right) = -\alpha (\gamma + 1) - (\gamma - 1) \]

\[ \beta = \frac{-\alpha (\gamma + 1) - (\gamma - 1)}{-\alpha (\gamma - 1) - (\gamma + 1)} = \frac{\alpha + \frac{\gamma - 1}{\gamma + 1}}{\alpha \frac{\gamma - 1}{\gamma + 1} + 1} \]

or

\[ \frac{p_*}{p_R} = \frac{p_*}{p_R} + \frac{\gamma - 1}{\gamma + 1} \]

\[ \frac{p_*}{p_R} \frac{\gamma - 1}{\gamma + 1} + 1 \]
Now we want the post shock velocity

\[ \hat{u}_+ = \hat{u}_R \cdot \frac{p_R}{\rho_R} \]

\[ u_+ - S = (u_R - S) \frac{p_R}{\rho_R} \]

\[ u_+ = \frac{p_R}{\rho_R} u_R + (1 - \frac{p_R}{\rho_R}) S \]

we had \[ S = u_R + c_F \left[ \left( \frac{\gamma+1}{2\gamma} \right) \left( \frac{p_F}{p_R} \right) + \frac{\gamma-1}{2\gamma} \right]^{\frac{1}{2}} \]

so

\[ u_+ = \frac{p_R}{\rho_R} u_R + \left( 1 - \frac{p_R}{\rho_R} \right) (u_R + A) \]

\[ = \frac{p_R}{\rho_R} u_R + u_R + A - \frac{p_R}{\rho_R} u_R - \frac{p_R}{\rho_R} A \]

\[ = u_R + A \left( 1 - \frac{p_R}{\rho_R} \right) \]

now

\[ 1 - \frac{p_R}{\rho_R} = 1 - \frac{P_R}{P_R} \frac{\gamma-1}{\delta+1} + 1 \]

\[ = \frac{P_R}{P_R} + \frac{\gamma-1}{\delta+1} - P_R \frac{\gamma-1}{\delta+1} + 1 \]

\[ = \frac{P_R}{P_R} + \frac{\gamma-1}{\delta+1} - \frac{P_R}{P_R} \frac{\gamma-1}{\delta+1} + 1 \]
\[
\frac{P_4}{P_R} \left( \frac{\gamma - 1}{\gamma + 1} \right) - \left( \frac{\gamma - 1}{\gamma + 1} \right) = \frac{P_4}{P_R} \frac{\gamma - 1}{\gamma + 1}
\]
\[
= \left( \frac{P_4}{P_R} - 1 \right) \left( 1 - \frac{\gamma - 1}{\gamma + 1} \right) = \frac{P_4}{P_R} - 1 \\frac{2}{\gamma + 1}
\]
\[
\frac{P_4}{P_R} \frac{\gamma - 1}{\gamma + 1}
\]
\[
\text{now putting this in our expression of } U_*
\]
\[
U_* = U_R + C_F \left[ \left( \frac{\gamma + 1}{2\gamma} \right) \left( \frac{P_4}{P_R} \right) + \frac{\gamma - 1}{2\gamma} \right] \left( \frac{P_4}{P_R} - 1 \right) \frac{2}{\gamma + 1}
\]
\[
\frac{P_4}{P_R} \frac{\gamma - 1}{\gamma + 1}
\]
\[
= U_R + \frac{2c_F}{\gamma + 1} \left( \frac{P_4}{P_R} - 1 \right) \left( \frac{\gamma + 1}{2\gamma} \right)^{1/2} \left( \frac{P_4}{P_R} + \frac{\gamma - 1}{\gamma + 1} \right)^{-1/2}
\]
\[
= U_R + \frac{2c_F}{(2\gamma)^{1/2}} \left( \frac{P_4}{P_R} - 1 \right) \frac{P_4}{P_R} \frac{\gamma - 1}{\gamma + 1} \left( \frac{P_4}{P_R} + \frac{\gamma - 1}{\gamma + 1} \right)^{-1/2}
\]
\[
= U_R - \frac{2c_F}{\sqrt{2\gamma(\gamma - 1)}} \left( \frac{\gamma + 1}{\gamma - 1} \right)^{1/2} \left( \frac{P_4}{P_R} + 1 \right)^{1/2}
\]
In steady state we know

\[ v \frac{dV}{dx} = - \frac{\gamma p}{\rho} \frac{dp}{dx} \quad \text{(Choudhuri Eq 6.63)} \]

Substituting \( \gamma = \frac{d\gamma}{dP} \)

\[ v \frac{dV}{dx} = - \frac{\gamma p}{\rho^2} \frac{dp}{dx} \]

Now assuming we are adiabatic,

\[ p = K \rho^\gamma \quad (K \text{ is a constant}) \]

Then

\[ v \frac{dV}{dx} = - \gamma K \rho^{\gamma-2} \frac{dp}{dx} \]

We want to integrate until stagnation \((v=0)\). We'll call the density at stagnation \( \rho_o \)

\[ \int_v^0 v' \, dv = - \int_{\rho}^{\rho_o} \gamma K \rho^{\gamma-2} \, dp' \]

\[ - \frac{1}{2} v^2 = - \gamma K \left[ \frac{1}{\gamma-1} \rho^{\gamma-1} \right]_p^{\rho_o} \]

\[ v^2 = \frac{2 \gamma K}{\gamma-1} \left[ \rho_o^{\gamma-1} - \rho^{\gamma-1} \right] \]

Not the \( \rho \) version

Convert to Mach number by dividing by \( \gamma = \frac{d\gamma}{dP} \)

\[ \left( \frac{v}{c_s} \right)^2 = \frac{2K}{\gamma-1} \frac{\rho}{\rho_o} \left[ \rho_o^{\gamma-1} - \rho^{\gamma-1} \right] = \frac{2}{\gamma-1} \left[ \left( \frac{\rho_o}{\rho} \right)^{\gamma-1} - 1 \right] \]

Finally, taking \( \rho_o^\gamma = \left( \frac{P_o}{P} \right)^{\gamma-1} \),

\[ M^2 = \frac{2}{\gamma-1} \left[ \left( \frac{P_o}{P} \right)^{\gamma-1} - 1 \right] \]
What about the area?

We know \( \rho v A = \text{const} \)

so \( A \sim \frac{1}{\rho v} \)

\[ v = c_s \cdot \frac{2}{8-1} \left[ \left( \frac{p_0}{p} \right)^{\frac{\gamma-1}{2}} - 1 \right] \]

so \( A \sim \frac{1}{\rho} \cdot \frac{\gamma-1}{2} \left( \frac{\rho}{\gamma p} \right)^{\frac{\gamma-1}{2}} \left[ \left( \frac{p_0}{p} \right)^{\frac{\gamma-1}{2}} - 1 \right]^{-\frac{1}{2}} \)

taking \( \rho \sim p^{1/3} \) and dropping constants

\[ A \sim \left( \frac{1}{p^{1/3}} \right)^{\frac{\gamma-1}{2}} \left[ \left( \frac{p_0}{p} \right)^{\frac{\gamma-1}{2}} - 1 \right]^{-\frac{1}{2}} \]

\[ \sim \left( \frac{p_0}{p} \right)^{\frac{\gamma-1}{2}} \left( 1 + \frac{1}{2} \right) \left[ \left( \frac{p_0}{p} \right)^{\frac{\gamma-1}{2}} - 1 \right]^{-\frac{1}{2}} \]

\( p_0 \) is constant, so we can put it here as normalization.