1. In class, we wrote down the momentum equation, derived from the Boltzmann equation as:

\[
\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i v_j) + \frac{\partial P_{ij}}{\partial x_i} = \rho g_i
\] (1)

where here we’ve taken the force, \( F_j \), to be the gravitational force, \( mg_j \).

Take the gravitational acceleration to be from the Poisson equation:

\[
g_j = -\frac{\partial \phi}{\partial x_j}; \quad \nabla^2 \phi = 4\pi G \rho \] (2)

and introduce the gravitational stress tensor:

\[
G_{ij} = -\frac{1}{4\pi G} \left( g_i g_j - \frac{1}{2} |\mathbf{g}|^2 \delta_{ij} \right) \] (3)

Show that by using the gravitational stress tensor, we can write the momentum equation in conservative form (e.g., time evolution in terms of the divergence of a flux with no explicit source terms):

\[
\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i v_j + P_{ij} - G_{ij}) = 0
\] (4)

2. (based on Shu) Define the specific entropy, \( s \), via:

\[
\rho s = -k_B \int f(x, v, t) \log f(x, v, t) d^3v
\] (5)

For a dilute gas in thermodynamic equilibrium, show that we can express

\[
s = c_v \log \left( \frac{p}{\rho^\gamma} \right) + \text{constant}
\] (6)

where \( c_v \) is the specific heat at constant volume,

\[
c_v = \frac{3 k_B}{2 m}
\] (7)

and \( \gamma = 5/3 \) is the ratio of specific heats for a monatomic ideal gas.

3. In class, we ignored the influence of an external force when finding the equilibrium distribution function of the Boltzmann equation for a dilute gas. Now consider that we have a non-negligible external field, \( \mathbf{F} \), obtained from a potential, \( \phi(x) \), as \( \mathbf{F} = -\nabla \phi(x) \). Show that the equilibrium distribution function in this case is

\[
f(x, v) = f_0(v) e^{-\phi(x)/kT}
\] (8)

where \( f_0(v) \) is the Maxwell-Boltzmann distribution.
Q. Show that the momentum equation with a gravitational source term can be written as

\[ \frac{\partial}{\partial t} (pu_i) + \frac{\partial}{\partial x_k} (pu_i u_k + p_i k - G_{ik}) = 0 \]

where \( G_{ik} \) is the "gravitational stress tensor",

\[ G_{ik} = -\frac{1}{4\pi G} \left( g_{ij} g_{lk} - \frac{1}{2} g^1 g_{ik} \right) \]

and the gravitational field satisfies Poisson's eq:

\[ \nabla^2 \phi = 4\pi G \rho \]

\[ \rho = -\nabla \phi \]

Let's look at

\[ \frac{\partial}{\partial x_k} G_{ik} = -\frac{1}{4\pi G} \frac{\partial}{\partial x_k} \left( g_{ij} g_{lk} - \frac{1}{2} g^1 g_{ik} \right) \]

\[ = -\frac{1}{4\pi G} \left[ g_{ij} \frac{\partial}{\partial x_k} g_{lk} + g_k \frac{\partial}{\partial x_k} g_{ij} - \frac{1}{2} \frac{\partial}{\partial x_i} g_{ik} \right] \]

\[ = -\frac{1}{4\pi G} \left[ g_{ij} \frac{\partial}{\partial x_k} g_{lk} + g_k \frac{\partial}{\partial x_k} g_{ij} - g_k \frac{\partial}{\partial x_i} g_{ik} \right] \]

now write \( g = -\nabla \phi \)

\[ = -\frac{1}{4\pi G} \left[ \frac{\partial \phi}{\partial x_i} \frac{\partial}{\partial x_k} \phi + \frac{\partial \phi}{\partial x_k} \frac{\partial}{\partial x_i} \phi - \frac{\partial}{\partial x_i} \phi \frac{\partial}{\partial x_k} \phi \right] \]

\[ = -\frac{1}{4\pi G} \frac{\partial \phi}{\partial x_i} \frac{\partial^2 \phi}{\partial x_k^2} \]

but \( \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \)

\[ \rho = -\nabla \phi \]

Poison's eq
\[ \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j + P_{ik}) = \frac{\partial}{\partial x_j} \sigma_{ik} = \rho g_i \]

that's our usual form of the momentum equation
Q. Define specific entropy as

\[ \rho s = - k_B \int f(\mathbf{r}, \mathbf{v}, t) \ln f(\mathbf{r}, \mathbf{v}, t) \, d^3v \]

show that for the dilute gas in thermodynamic equilibrium,

\[ s \sim c_v \ln \left( \frac{P}{\rho s} \right) \]

with \( c_v = \frac{3k}{2m} \) and \( \gamma = \frac{5}{3} \)

We have \( f = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} \) in equilibrium so

\[ \rho s = - k_B n \left( \frac{m}{2\pi kT} \right)^{3/2} \int d^3v \, e^{-\frac{mv^2}{2kT}} \ln \left[ n \left( \frac{m}{2\pi kT} \right)^{3/2} - \frac{mv^2}{2kT} \right] \]

expanding the log,

\[ \rho s = - k_B n \left( \frac{m}{2\pi kT} \right)^{3/2} \left\{ \int d^3v \, \frac{\partial}{\partial v^2} \ln \left[ \left( \frac{m}{2\pi kT} \right)^{3/2} - \frac{mv^2}{2kT} \right] e^{-\frac{mv^2}{2kT}} + \int d^3v \, \left( - \frac{mv^2}{2kT} \right) e^{-\frac{mv^2}{2kT}} \right\} \]

Let's look at the two integrals separately.

Taking \( \alpha = \frac{m}{2kT} \)

\[ i: \int d^3v \, \ln \left( \frac{n^{3/2} \alpha}{\pi} \right)^{3/2} e^{-\alpha v^2} \]

\[ = \frac{3}{2} \ln \left( \frac{n^{3/2} \alpha}{\pi} \right) 4\pi \int_0^{\infty} v^2 e^{-\alpha v^2} \, dv = \]

\[ = \frac{3}{2} \ln \left( \frac{n^{3/2} \alpha}{\pi} \right) 4\pi \int_0^{\infty} v^2 e^{-\alpha v^2} \, dv \]
\[
\rho_S = -k_B n \left( \frac{\alpha}{\pi} \right)^{3/2} \left[ \frac{3}{2} \ln \left( \frac{n^{3/2} \alpha}{\pi} \right) - \frac{3}{2} \frac{3}{2} \alpha^{-3/2} \right] \\
\rho_S = -k_B n \frac{3}{2} \left[ \ln \left( \frac{n^{3/2} \alpha}{\pi} \right) - 1 \right]
\]

\text{now}
\[
s = -\frac{1}{\rho} k_B n \frac{3}{2} \left[ \ln \left( \frac{n^{3/2} \alpha}{\pi} \right) - 1 \right] \\
\frac{1}{\rho} = \frac{1}{nm}
\]
\[
s = -c_v \left[ \ln \left( \frac{n^{3/2} \alpha}{\pi} \right) - 1 \right]
\]
\[
c_v = \frac{8k_B}{2m}
\]
\[
s = -c_v \left[ \ln \left( \frac{n^{3/2} m}{2k_B T} \right) + C \right] \\
\downarrow \text{const}
\]
\[
s = c_v \ln \left( \frac{2k_B T}{m n^{3/2}} \right) + \text{const}
\]
Using \( p = n k T \)

\[
\frac{p}{p_0^{5/3}} = \frac{nkT}{(mn)^{5/3}} = \frac{kT}{n^{5/3}} \cdot n^{-5/3}
\]

\[
\therefore \text{ up to constants,}
\]

\[
s = c_v \ln \left( \frac{p}{p_0} \right) + \text{const}
\]
Q. In class we ignored the influence of an external forcing when finding the equilibrium distribution function for a dilute gas.

Now consider we have an external field \( \mathbf{F} \) specified from a potential \( \phi(x) \). Show that the equilibrium solution is

\[
f(x, \mathbf{v}) = f_0(\mathbf{v}) e^{-\phi(x)/kT} \quad (\mathbf{F} = -\nabla \phi)
\]

where \( f_0(\mathbf{v}) \) is the Maxwell–Boltzmann distribution.

By definition the collision term vanishes in equilibrium, so we need to show

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial x} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0
\]

Substituting in our expression:

\[
\frac{\partial f}{\partial t} = 0 \quad (\text{there is no time dependence in equilibrium})
\]

\[
\mathbf{v} \cdot \frac{\partial f}{\partial x} = \mathbf{v} \cdot f_0 \cdot \frac{\partial}{\partial x} e^{-\phi(x)/kT}
\]

\[
= -\frac{\partial}{\partial x} \left( f_0 \mathbf{v} \cdot \nabla \phi \right) = \frac{\mathbf{v} \cdot \mathbf{F}}{kT} f_0
\]

\[
\frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}} \left[ \frac{m}{2\pi kT} e^{-\frac{m}{2kT} (\mathbf{v} - \mathbf{v}_0)^2} \right]
\]

\[
= -\frac{mv}{kT} \frac{\mathbf{F}}{m} f_0
\]

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial x} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\mathbf{v} \cdot \mathbf{F}}{kT} f_0 - \frac{\mathbf{v} \cdot \mathbf{F}}{kT} f_0 = 0
\]