1. In this problem, we will write the momentum equation in a different form, which we will use in class to demonstrate some new properties of the Euler equations.

(a) Prove the vector identity:
\[(\mathbf{v} \cdot \nabla) \mathbf{v} = (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{1}{2} \nabla |\mathbf{v}|^2\]  

where \(\mathbf{v}\) is any arbitrary vector.

*Hint:* It may be best to work in index notation, and recall that
\[\nabla \times \mathbf{v} = \epsilon_{ijk} \partial_j v_k\]
where \(\epsilon_{ijk}\) is the permutation (or Levi-Civita) tensor with the following properties:
\[\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji}\]
\[\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\]

(b) Using this identity, rewrite the momentum equation as:
\[\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} |\mathbf{v}|^2 \right) + (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{1}{\rho} \nabla p = \mathbf{g}\]  

2. The internal energy evolution equation takes the form:
\[\rho \frac{De}{Dt} + p \nabla \cdot \mathbf{v} = 0\]

(in the absence of heat sources). Here we want to look at evolution equations for other thermodynamic quantities that might take the place of energy.

(a) Derive an equation for the evolution of temperature of the form:
\[c_v \frac{DT}{Dt} = \ldots\]  

*Hint,* starting with the internal energy equation, expressing \(e = e(\rho, T)\) and note that the specific heat at constant volume is \(c_v = \frac{\partial e}{\partial T} \bigg|_\rho\).

(b) Derive an equation for the evolution of pressure of the form:
\[\frac{Dp}{Dt} = \ldots\]  

Do not assume an ideal or gamma-law gas. It might be easiest to express \(p = p(\rho, s)\) and to note that the first adiabatic index is \(\Gamma_1 = \frac{\partial \log p}{\partial \log \rho} \bigg|_s\).

(c) Derive an equation for the evolution of specific enthalpy, \(h = e + p/\rho\) of the form:
\[\rho \frac{Dh}{Dt} = \ldots\]  

Under what conditions is enthalpy a conserved quantity?
3. (Choudhuri 4.2) Suppose an object heavier than water is fully immersed in water within which pressure varies as given by (4.30) \( p = p_0 - \rho g z \). Show that the net force exerted on the object by the surrounding water is \(-M'g\), where \( M' \) is the mass of water displaced by the object. This is the celebrated *Archimedes’s principle*. 