1. In class, we wrote down the momentum equation, derived from the Boltzmann equation as:

\[
\frac{\partial \rho v_j}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i v_j) + \frac{\partial P_{ij}}{\partial x_i} = \rho g_j
\]  

(1)

where here we’ve taken the force, \( F_j \) to be the gravitational force, \( mg_j \).

Take the gravitational acceleration to be from the Poisson equation:

\[
\begin{align*}
g_j &= -\frac{\partial \phi}{\partial x_j} \\
\nabla^2 \phi &= 4\pi G \rho
\end{align*}
\]  

(2)

and introduce the gravitational stress tensor:

\[
G_{ij} = -\frac{1}{4\pi G} \left( g_i g_j - \frac{1}{2} |g|^2 \delta_{ij} \right)
\]  

(3)

Show that by using the gravitational stress tensor, we can write the momentum equation in conservative form (e.g., time evolution in terms of the divergence of a flux with no explicit source terms):

\[
\frac{\partial \rho v_j}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i v_j + P_{ij} - G_{ij}) = 0
\]  

(4)

2. (based on Shu) Define the specific entropy, \( s \), via:

\[
\rho s = -k_B \int f(x, v, t) \log f(x, v, t) d^3v
\]  

(5)

For a dilute gas in thermodynamic equilibrium, show that we can express

\[
s = c_v \log \left( \frac{P}{\rho \gamma} \right) + \text{constant}
\]  

(6)

where \( c_v \) is the specific heat at constant volume,

\[
c_v = \frac{3}{2} \frac{k_B}{m}
\]  

(7)

and \( \gamma = 5/3 \) is the ratio of specific heats for a monatomic ideal gas.

3. In class, we ignored the influence of an external force when finding the equilibrium distribution function of the Boltzmann equation for a dilute gas. Now consider that we have a non-negligible external field, \( \mathbf{F} \), obtained from a potential, \( \phi(x) \), as \( \mathbf{F} = -\nabla \phi(x) \). Show that the equilibrium distribution function in this case is

\[
f(x, v) = f_0(v) e^{-\phi(x)/kT}
\]  

(8)

where \( f_0(v) \) is the Maxwell-Boltzmann distribution.